M337

Complex analysis

Book A

Complex numbers and functions

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Unit A1

Complex numbers

Introduction

Historical survey

Before describing the contents of this unit, we provide a brief historical account of the development of complex numbers.

Many mathematical problems that have been studied since ancient times lead to quadratic equations of the form

$$ax^2 + bx + c = 0, \qquad (0.1)$$

where a, b, c are real numbers, with $a \neq 0$, and x is an unknown number. The formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

for the solutions of equation (0.1) gives

two distinct real solutions if $b^2 - 4ac > 0$

one (repeated) real solution if $b^2 - 4ac = 0$

no real solutions if $b^2 - 4ac < 0$.

When $b^2 - 4ac < 0$, equation (0.1) has no real solutions because a negative number cannot have a real square root.

However, as long ago as the sixteenth century, Italian scholars such as Gerolamo Cardano (1501–1576) began to experiment with the manipulation of symbols such as $\sqrt{-1}$, using the ordinary rules for real numbers. For example, he considered the problem of finding numbers x and y such that

$$x + y = 10 \quad \text{and} \quad xy = 40. \qquad (0.2)$$

This problem has no real solutions, since if we substitute $y = 40/x$ into $x + y = 10$, then we obtain the quadratic equation

$$x^2 - 10x + 40 = 0,$$

which has no real solutions (because $b^2 - 4ac = -60 < 0$).

Cardano pointed out, however, that if $\sqrt{-15}$ is manipulated using the ordinary rules for real numbers, then

$$x = 5 + \sqrt{-15} \quad \text{and} \quad y = 5 - \sqrt{-15}$$

do satisfy equations (0.2). As he wrote:

> Putting aside the mental tortures involved, multiply $5 + \sqrt{-15}$ by $5 - \sqrt{-15}$, making $25 - (-15)$, whence the product is 40.
>
> (Burton, 2002, p. 297)

At the time there was little enthusiasm for such a solution because of doubts about the existence of entities such as $\sqrt{-15}$, doubts which Cardano himself harboured, but soon the idea of taking square roots of negative numbers was to prove its worth in a significant way, which we now describe.

The general cubic equation

$$ax^3 + bx^2 + cx + d = 0, \tag{0.3}$$

where a, b, c, d are real numbers, with $a \neq 0$, is much more difficult to solve algebraically than is the quadratic equation. One of the first to find real-number solutions to general cubic equations was the Persian mathematician (and poet) Omar Khayyām (1048–1131), who made ingenious use of conic sections. However, only after the work of Cardano and his contemporaries in the sixteenth century did mathematicians begin to appreciate that there are other, non-real solutions of cubic equations. Their breakthrough is described by the following remarkable method, which is usually attributed to Scipione del Ferro (1465–1526) and Niccolò Fontana Tartaglia (c.1500–1557).

First, equation (0.3) is reduced to the form

$$x^3 + px + q = 0 \tag{0.4}$$

by substituting $x - \frac{1}{3}(b/a)$ in place of x and dividing through by a. We will now solve this equation; there is no need to follow the details if you do not wish to. Substitute

$$x = u + v, \quad \text{where } uv = -p/3,$$

assuming that such real numbers u and v exist. This transforms equation (0.4) into

$$(u + v)^3 + p(u + v) + q = 0,$$

which, on expanding the cubic term, gives

$$u^3 + 3uv(u + v) + v^3 + p(u + v) + q = 0.$$

Since $uv = -p/3$, this reduces to

$$u^3 + v^3 + q = 0, \tag{0.5}$$

and multiplying through by u^3, we obtain

$$u^6 + qu^3 - (p/3)^3 = 0.$$

This is a quadratic equation in u^3 with solutions

$$u^3 = \frac{-q \pm \sqrt{q^2 + 4(p/3)^3}}{2}.$$

Then, by equation (0.5),

$$v^3 = \frac{-q \mp \sqrt{q^2 + 4(p/3)^3}}{2},$$

so one of the solutions of equation (0.4) is

$$x = \sqrt[3]{\frac{-q + \sqrt{q^2 + 4(p/3)^3}}{2}} + \sqrt[3]{\frac{-q - \sqrt{q^2 + 4(p/3)^3}}{2}}. \tag{0.6}$$

Formula (0.6) works extremely well in some cases. For example, the equation

$$x^3 + 3x - 4 = 0$$

has $p = 3$ and $q = -4$, so it follows from equation (0.6) that

$$x = \sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}}.$$

Now, one can check that

$$\left(\tfrac{1}{2}(1+\sqrt{5})\right)^3 = 2+\sqrt{5} \quad \text{and} \quad \left(\tfrac{1}{2}(1-\sqrt{5})\right)^3 = 2-\sqrt{5},$$

so it follows that

$$x = \tfrac{1}{2}(1+\sqrt{5}) + \tfrac{1}{2}(1-\sqrt{5}) = 1,$$

which is indeed a solution of $x^3 + 3x - 4 = 0$.

For some values of p and q, though, there is a difficulty. The Italian mathematician Rafael Bombelli (1526–1572), who was probably the first mathematician bold enough to accept the existence of square roots of negative numbers, considered the equation

$$x^3 - 15x - 4 = 0. \tag{0.7}$$

This has $p = -15$ and $q = -4$, so it follows from equation (0.6) that

$$x = \sqrt[3]{2+\sqrt{-121}} + \sqrt[3]{2-\sqrt{-121}}.$$

This solution involves the expression $\sqrt{-121}$, which suggests that, for this cubic equation, formula (0.6) does not give a solution. However, following Bombelli and treating $\sqrt{-1}$ in the same way as a real number, we see that

$$(\sqrt{-1})^3 = (\sqrt{-1})^2 \times \sqrt{-1} = -\sqrt{-1},$$

and hence

$$\begin{aligned}(2+\sqrt{-1})^3 &= 2^3 + (3\times 2^2 \times \sqrt{-1}) + (3 \times 2 \times (\sqrt{-1})^2) + (\sqrt{-1})^3 \\ &= 8 + 12\sqrt{-1} - 6 - \sqrt{-1} \\ &= 2 + 11\sqrt{-1} \\ &= 2 + \sqrt{-121}.\end{aligned}$$

Similarly,

$$(2-\sqrt{-1})^3 = 2 - \sqrt{-121}.$$

Hence

$$x = (2+\sqrt{-1}) + (2-\sqrt{-1}) = 4,$$

which is indeed a solution of equation (0.7). Thus, by allowing the use of symbols that seemingly have no meaning, we can produce one correct solution to the original problem.

This method of solution did not immediately lead to acceptance of such 'imaginary numbers' (as they were called), which continued to be regarded with great suspicion. A century later, for example, the English mathematician and scientist Isaac Newton (1642–1727) stated that if the solution to a problem involved imaginary numbers, then the problem did not have a 'genuine' solution.

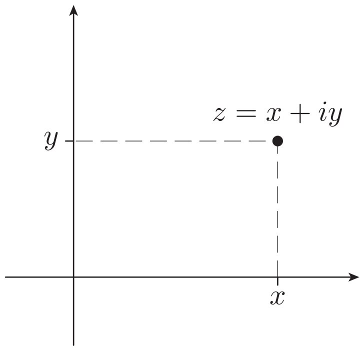


Figure 0.1 Representation of $x + iy$ in the complex plane

By the eighteenth century, however, mathematicians such as Johann Bernoulli (1667–1748), Gottfried Wilhelm Leibniz (1646–1716) and Leonhard Euler (1707–1783) were using imaginary numbers in applications to integration.

The symbol i for $\sqrt{-1}$ was introduced by Euler, and the name *complex number* for an expression of the form $z = x + iy$, where x and y are real, was introduced by the German mathematician Carl Friedrich Gauss (1777–1855) at the end of the eighteenth century to replace the old phrase 'imaginary number'. Gauss also advocated the geometric interpretation of a complex number $x + iy$ as a point with rectangular coordinates (x, y) in a plane (see Figure 0.1).

The importance of complex numbers was underlined by the so-called Fundamental Theorem of Algebra, which was established in the late eighteenth and early nineteenth centuries. This theorem states that every polynomial equation

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0, \tag{0.8}$$

where $a_0, a_1, \ldots, a_n$ are complex numbers, with $a_n \neq 0$, and z is an unknown complex number, has at least one complex solution (and hence n complex solutions, some of which may be repeated). Thus, although it was necessary to introduce the new 'complex' numbers in order to solve quadratic equations, there was no need to introduce any further new numbers in order to solve *all* polynomial equations with complex coefficients. We should hasten to add that the mathematicians of the eighteenth and nineteenth centuries did not find a general formula for solving equation (0.8) for all values of n; indeed, in 1824 the Norwegian mathematician Niels Henrik Abel (1802–1829) proved that for $n \geq 5$ no formula that uses the usual operations of arithmetic exists. Nonetheless, the Fundamental Theorem of Algebra asserts the *existence* of a solution.

Any lingering doubts about the validity of complex numbers were laid to rest in the 1830s when the Irish mathematician William Rowan Hamilton (1805–1865) gave a definition of complex numbers as ordered pairs of real numbers, writing (a, b) in place of $a + ib$, subject to the rules of manipulation

$$(a, b) + (c, d) = (a + c, b + d),$$
$$(a, b) \times (c, d) = (ac - bd, ad + bc).$$

This had the effect of placing complex numbers on a sound algebraic basis.

The groundwork had now been laid for the study of complex numbers to flourish. This development, associated with the names of Cauchy, Riemann and Weierstrass, went on throughout the nineteenth century.

Although most of this module deals with the classical theory of complex numbers and complex functions, at the end of the module you will see that even today there are still new and exciting developments in the subject.

Applications, supplementary topics and history

Scattered throughout the module you will find boxes like this one that contain additional information about complex analysis, to supplement the text. *The content of these boxes is not assessed.* Some of the boxes are about applications of complex analysis to science and engineering, and others describe further branches of mathematics related to complex analysis. You will also find out about the history of some of the mathematicians who have shaped the subject, some of whom you have encountered in this Introduction already.

Introduction to this unit

The first book of the module is an introduction to complex functions, and it is designed to familiarise you with their most basic properties. This unit is devoted solely to complex numbers themselves.

In Section 1 we define complex numbers and show you how to manipulate them, stressing the similarities with the manipulation of real numbers.

Section 2 is about the geometric representation of complex numbers. You will find that this is useful in understanding the arithmetic properties introduced in the first section.

In Section 3 we discuss methods of finding nth roots of complex numbers and the solutions of simple polynomial equations.

The final two sections deal with inequalities between real-valued expressions involving complex numbers. First we use inequalities in Section 4 to describe various subsets of the complex plane. Then in Section 5 we introduce the Triangle Inequality, which is a useful tool for manipulating inequalities.

Each section ends with a number of further exercises for additional practice. (This is the case for most sections throughout the module.)

Unit guide

After studying this unit, you should be able to perform basic algebraic manipulations with complex numbers and understand their geometric interpretation. Before you tackle later units, it is important that you become confident with these basic manipulations, and you should attempt as many of the exercises as you have time for.

1 Complex numbers and their properties

After working through this section, you should be able to:

- determine the *real part*, the *imaginary part* and the *complex conjugate* of a given *complex number*
- perform addition, subtraction, multiplication and division of complex numbers
- use the Binomial Theorem and the Geometric Series Identity to simplify complex expressions.

1.1 Defining complex numbers

We assume that you are already familiar with various different types of numbers, such as the **natural numbers** $\mathbb{N} = \{1, 2, 3, \ldots\}$, the **integers** $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the **rational numbers** (or fractions) $\mathbb{Q} = \{p/q : p \in \mathbb{Z}, q \in \mathbb{N}\}$, and the **real numbers** $\mathbb{R}$, which can be represented by decimals, either terminating (such as $\frac{1}{2} = 0.5$) or non-terminating (such as $\pi = 3.1415\ldots$). We assume also that you are familiar with the usual arithmetic operations of addition, subtraction, multiplication and division of real numbers.

We are now going to introduce the idea of a complex number, and we begin with some definitions.

Definitions

A **complex number** z is an expression of the form $x + iy$, where x and y are real numbers and i is a symbol with the property that $i^2 = -1$. We write

$$z = x + iy \quad \text{or, equivalently,} \quad z = x + yi,$$

and say that z is expressed in **Cartesian form**. The real number x is the **real part** of z (written $x = \operatorname{Re} z$) and the real number y is the **imaginary part** of z (written $y = \operatorname{Im} z$).

Two complex numbers are **equal** if their real parts are equal *and* their imaginary parts are equal.

The set of all complex numbers is denoted by $\mathbb{C}$.

In this module, the mathematical symbol z always denotes a complex number, unless specified otherwise.

The word 'Cartesian' is derived from the surname of the French mathematician and philosopher René Descartes (1596–1650), who was a pioneer in relating algebra to geometry by representing pairs of real variables x and y by points in a plane.

The following table gives some examples of complex numbers z that correspond to given real numbers x and y.

$z = x + iy$	$1+2i$	$\sqrt{2}+i\pi$	$3i$	1	$1+i$	0	$1-2i$
$\operatorname{Re} z = x$	1	$\sqrt{2}$	0	1	1	0	1
$\operatorname{Im} z = y$	2	π	3	0	1	0	-2

When working with complex numbers we use the following conventions, some of which you can see in the table.

- Any real number x can be thought of as a complex number whose imaginary part is zero (thus $\mathbb{R}$ is a subset of $\mathbb{C}$). We write, for example, $1 + 0i = 1$.
- If the real part of a complex number is 0, but the imaginary part is non-zero, then we omit the real part when writing the complex number; for example, $0 + 3i = 3i$.
- The complex number $0 + 0i$ is written 0, the zero complex number.
- We usually abbreviate $1i$ to i and $-1i$ to $-i$.
- If y is negative, then we usually write z as $x - |y|i$; for example, $1 + (-2)i = 1 - 2i$.

In some texts, a complex number with imaginary part zero is called *purely real*, and a complex number with real part zero is called *purely imaginary*. However, those terms will not be used in this module. We also remark that in certain contexts such as electrical engineering (where i is used for current) it is common practice to write j instead of i.

1.2 Arithmetic with complex numbers

The definition of a complex number contains the symbol '+' and refers to the 'square' of i. This suggests that arithmetic operations can be performed with complex numbers; the following definitions are made.

Definitions

The binary operations of **addition**, **subtraction** and **multiplication** of complex numbers are denoted by the same symbols as for real numbers and are performed by the usual procedure – that is, treating complex numbers as real expressions together with an algebraic symbol i with the property that $i^2 = -1$.

The following example shows some arithmetic operations involving complex numbers.

Example 1.1

Express each of the following numbers in Cartesian form.

(a) $(1+2i)+\left(\frac{1}{2}+\pi i\right)$

(b) $(1+2i)\left(\frac{1}{2}+\pi i\right)$

(c) $2(1+2i)-2i\left(\frac{1}{2}+\pi i\right)$

(d) $(1+2i)(1-2i)$

Solution

(a) By the usual procedure,

$$\begin{aligned}(1+2i)+\left(\tfrac{1}{2}+\pi i\right) &= 1+2i+\tfrac{1}{2}+\pi i\\ &= \tfrac{3}{2}+(2+\pi)i.\end{aligned}$$

(b) By the usual procedure,

$$(1+2i)\left(\tfrac{1}{2}+\pi i\right)=\tfrac{1}{2}+\pi i+i+2\pi i^2.$$

Applying the extra property that $i^2=-1$, we obtain

$$(1+2i)\left(\tfrac{1}{2}+\pi i\right)=\left(\tfrac{1}{2}-2\pi\right)+(\pi+1)i.$$

(c) By the usual procedure and the property that $i^2=-1$,

$$\begin{aligned}2(1+2i)-2i(\tfrac{1}{2}+\pi i) &= 2+4i-i-2\pi i^2\\ &= (2+2\pi)+3i.\end{aligned}$$

(d) By the usual procedure and the property that $i^2=-1$,

$$\begin{aligned}(1+2i)(1-2i) &= 1-2i+2i-4i^2\\ &= 1+4\\ &= 5.\end{aligned}$$

The following exercises provide practice at manipulating complex numbers.

Exercise 1.1

(a) Express each of the following numbers in Cartesian form.

(i) $(2+i)+3i(-1+3i)$

(ii) $(2+i)(-1+3i)$

(iii) $(-1+3i)(-1-3i)$

(b) Write down the real and imaginary parts of $z=(2+i)+3i(-1+3i)$.

In the next exercise the letters x and y appear in the specification of complex numbers (with and without subscripts). You should assume in these circumstances – here and elsewhere in the module – that x and y are both real numbers.

Exercise 1.2

Express each of the following numbers in Cartesian form.

(a) $(x_1 + iy_1) + (x_2 + iy_2)$ (b) $(x_1 + iy_1) - (x_2 + iy_2)$

(c) $(x_1 + iy_1)(x_2 + iy_2)$ (d) $(x + iy)(x - iy)$

As with real numbers, the negative $-z$ of a complex number z is defined in such a way that $z + (-z) = 0$.

Definition

The **negative** $-z$ of a complex number $z = x + iy$ is

$$-z = (-x) + i(-y),$$

usually written $-z = -x - iy$.

For example, $-(1 + i) = -1 - i$.

Next we discuss division of complex numbers. As with real numbers, the reciprocal $1/z$ of a non-zero complex number z is defined in such a way that $z(1/z) = 1$.

Definitions

The **reciprocal** $1/z$ of a non-zero complex number $z = x + iy$ is

$$\frac{1}{z} = \frac{x - iy}{x^2 + y^2}.$$

The alternative notation z^{-1} is also used for the reciprocal.

The **quotient** z_1/z_2 of a complex number z_1 by a non-zero complex number z_2 is

$$\frac{z_1}{z_2} = z_1\left(\frac{1}{z_2}\right).$$

This definition of $1/z$ works because, as you saw in Exercise 1.2(d),

$$(x + iy)(x - iy) = x^2 + y^2,$$

which is strictly positive because z is non-zero, so

$$\begin{aligned} z\left(\frac{1}{z}\right) &= (x + iy)\left(\frac{x - iy}{x^2 + y^2}\right) \\ &= \frac{(x + iy)(x - iy)}{x^2 + y^2} \\ &= \frac{x^2 + y^2}{x^2 + y^2} = 1. \end{aligned}$$

The definition of quotient suggests that in order to evaluate z_1/z_2, we must first evaluate $1/z_2$ and then multiply by z_1. In practice, it is easier to do both operations at once using the following strategy.

Strategy for obtaining a quotient

To obtain the quotient

$$\frac{x_1 + iy_1}{x_2 + iy_2}, \quad \text{where } y_2 \neq 0,$$

in Cartesian form, multiply both numerator and denominator by $x_2 - iy_2$, so that the denominator becomes real.

Example 1.2

Express the following numbers in Cartesian form.

(a) $\dfrac{1}{1+2i}$ (b) $\dfrac{3+4i}{1+2i}$

Solution

(a) By the strategy for obtaining a quotient,

$$\begin{aligned}\frac{1}{1+2i} &= \frac{1-2i}{(1+2i)(1-2i)}\\ &= \frac{1-2i}{1+4} = \frac{1}{5} - \frac{2}{5}i.\end{aligned}$$

(b) By the strategy for obtaining a quotient,

$$\begin{aligned}\frac{3+4i}{1+2i} &= \frac{(3+4i)(1-2i)}{(1+2i)(1-2i)}\\ &= \frac{3-6i+4i-8i^2}{1+4}\\ &= \frac{11-2i}{5} = \frac{11}{5} - \frac{2}{5}i.\end{aligned}$$

It would be acceptable to leave the solution to part (b) in the form $(11-2i)/5$, since this can readily be reduced to Cartesian form.

Exercise 1.3

(a) Express the following numbers in Cartesian form.

(i) $\dfrac{1}{i}$ (ii) $\dfrac{1}{1+i}$ (iii) $\dfrac{1+2i}{2+3i}$

(b) Express the following quotient in Cartesian form:

$$\frac{x_1 + iy_1}{x_2 + iy_2}, \quad \text{where } y_2 \neq 0.$$

The process of changing the sign of the imaginary part of a complex number, used in the strategy above, is often carried out, so the following terminology and notation is helpful.

Definition

The **complex conjugate** $\overline{z}$ of a complex number $z = x + iy$ is

$$\overline{z} = x - iy.$$

Some texts use the notation z^* in place of $\overline{z}$. In particular, complex numbers feature considerably in the subject of quantum mechanics, and most texts on that subject prefer this alternative notation.

The complex conjugate of z satisfies the simple identities

$$\operatorname{Re}\overline{z} = \operatorname{Re} z \quad \text{and} \quad \operatorname{Im}\overline{z} = -\operatorname{Im} z.$$

Several more identities involving complex conjugates are given in the following result.

Theorem 1.1 Properties of the complex conjugate

(a) If z is a complex number, then

(i) $z + \overline{z} = 2\operatorname{Re} z$

(ii) $z - \overline{z} = 2i\operatorname{Im} z$

(iii) $\overline{(\overline{z})} = z$.

(b) If z_1 and z_2 are complex numbers, then

(i) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(ii) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

(iii) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

(iv) $\overline{z_1/z_2} = \overline{z_1}/\overline{z_2}$, where $z_2 \neq 0$.

Part (b)(i) says that 'the conjugate of a sum is the sum of the conjugates', and other parts can be described in similar terms. Note the use of the long conjugate bar over expressions involving several symbols.

Proof

(a) If $z = x + iy$, then $\overline{z} = x - iy$, so

(i) $z + \overline{z} = (x + iy) + (x - iy) = 2x = 2\operatorname{Re} z$

(ii) $z - \overline{z} = (x + iy) - (x - iy) = 2iy = 2i\operatorname{Im} z$

(iii) $\overline{(\overline{z})} = \overline{(x - iy)} = x + iy = z$.

(b) The proofs of these identities all follow from the results of Exercises 1.2 and 1.3(b). To illustrate the method, we prove part (iii).

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then, by Exercise 1.2(c),

$$z_1 z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2). \tag{1.1}$$

Now, $\overline{z_1} = x_1 - iy_1$ and $\overline{z_2} = x_2 - iy_2$, so if we replace y_1, y_2 by $-y_1, -y_2$ in equation (1.1), then we see that

$$\begin{aligned}\overline{z_1}\,\overline{z_2} &= (x_1x_2 - (-y_1)(-y_2)) + i(x_1(-y_2) + (-y_1)x_2)\\ &= (x_1x_2 - y_1y_2) - i(x_1y_2 + y_1x_2)\\ &= \overline{z_1z_2},\end{aligned}$$

as required. ■

Exercise 1.4

Prove the identities stated in Theorem 1.1(b), parts (i) and (iv).

Now that you have seen how to perform the usual arithmetic operations with complex numbers, it is natural to ask the following question. Do these operations have the usual properties that are known to hold for real numbers? It is a straightforward matter to check that, for example, addition of complex numbers is associative; that is, for all z_1, z_2, z_3 in $\mathbb{C}$,

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3).$$

It is also straightforward, but more tedious, to show that multiplication of complex numbers is associative; that is, for all z_1, z_2, z_3 in $\mathbb{C}$,

$$(z_1z_2)z_3 = z_1(z_2z_3).$$

In fact, it turns out that all the usual arithmetic properties do hold for complex numbers. These are summarised in the following table.

Arithmetic in $\mathbb{C}$

Property	Addition	Multiplication
Closure	**A1** For all z_1, z_2 in $\mathbb{C}$, $z_1 + z_2 \in \mathbb{C}$.	**M1** For all z_1, z_2 in $\mathbb{C}$, $z_1z_2 \in \mathbb{C}$.
Identity	**A2** For all z in $\mathbb{C}$, $z + 0 = 0 + z = z$.	**M2** For all z in $\mathbb{C}$, $z1 = 1z = z$.
Inverse	**A3** For all z in $\mathbb{C}$, $z + (-z) = (-z) + z = 0$.	**M3** For all non-zero z in $\mathbb{C}$, $zz^{-1} = z^{-1}z = 1$.
Associative	**A4** For all z_1, z_2, z_3 in $\mathbb{C}$, $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$.	**M4** For all z_1, z_2, z_3 in $\mathbb{C}$, $(z_1z_2)z_3 = z_1(z_2z_3)$.
Commutative	**A5** For all z_1, z_2 in $\mathbb{C}$, $z_1 + z_2 = z_2 + z_1$.	**M5** For all z_1, z_2 in $\mathbb{C}$, $z_1z_2 = z_2z_1$.
Distributive	**D** For all z_1, z_2, z_3 in $\mathbb{C}$, $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$.	

Once all these properties have been proved (and we will not give the details here), then the contents of the table can be described in algebraic terms as follows.

The complex numbers $\mathbb{C}$ form an abelian group under the operation of *addition*, with identity 0 (properties A1–A5).

The set of non-zero complex numbers is an abelian group under the operation of *multiplication* (noting in M1 that if $z_1, z_2 \neq 0$, then $z_1 z_2 \neq 0$), with identity 1 (properties M1–M5).

These two structures are linked by the *distributive* property (D).

Because $\mathbb{C}$ has all these properties, it is called a **field**. The rational numbers $\mathbb{Q}$ and the real numbers $\mathbb{R}$ are also fields.

Notice that in property M3 we have used z^{-1} to denote the reciprocal $1/z$. It is also standard practice to use the notation z^n, where $n \in \mathbb{Z}$, for integral powers of a non-zero complex number z. For example,

$$i^2 = -1, \quad i^3 = -i, \quad i^4 = 1,$$
$$i^{-1} = -i, \quad i^{-2} = -1, \quad i^{-3} = i, \quad i^{-4} = 1.$$

By convention, $z^0 = 1$, for all non-zero z, and 0^0 is *not* defined, except in special cases such as formulas in which it is convenient to assign a value to 0^0 (for example, see binomial coefficients in the next subsection). The zero complex number has powers $0^n = 0$ for $n = 1, 2, 3, \ldots$. We will discuss the meaning of fractional powers, such as $z^{1/2} = \sqrt{z}$, in Section 3, and also in Unit A2.

Rafael Bombelli

The Italian mathematician Rafael Bombelli, whom we met in the Introduction when discussing his contributions to solving cubic equations, is recognised as the first person to work with complex numbers in a systematic way. In his celebrated text *Algebra*, published in 1572, Bombelli lays out many of the rules for manipulating complex numbers that we have covered in this subsection, but with different terminology and notation. That text also includes an exposition of the solution of cubic and quartic equations, making use of complex numbers.

1.3 Identities with complex numbers

Because complex numbers satisfy the usual arithmetic properties, we can prove and then use all the usual algebraic identities. For example, if z_1 and z_2 are any complex numbers, then

$$(z_1 + z_2)^2 = z_1^2 + 2z_1z_2 + z_2^2 \quad \text{and} \quad z_1^2 - z_2^2 = (z_1 - z_2)(z_1 + z_2).$$

Thus, for example, if $z^2 + 9 = 0$, then

$$z^2 + 9 = (z - 3i)(z + 3i) = 0,$$

so $z = 3i$ or $z = -3i$.

Exercise 1.5

Prove the following identities.

(a) $(z_1 + z_2)^3 = z_1^3 + 3z_1^2 z_2 + 3z_1 z_2^2 + z_2^3$

(b) $z_1^3 - z_2^3 = (z_1 - z_2)(z_1^2 + z_1 z_2 + z_2^2)$

(c) $z_1^3 + z_2^3 = (z_1 + z_2)(z_1^2 - z_1 z_2 + z_2^2)$

The identities in Exercise 1.5 are, in fact, special cases of two important general identities which will often be used in the module. The first of these is the Binomial Theorem, which we state in two forms. This theorem uses the binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!},$$

sometimes written as ${}^nC_k$. Note that, by convention, $0! = 1$ and $0^0 = 1$ in the formulas in these identities.

The proof of the Binomial Theorem is the same as in the real case, so it is omitted.

Theorem 1.2 Binomial Theorem

(a) If $z \in \mathbb{C}$ and $n \in \mathbb{N}$, then

$$\begin{aligned}(1+z)^n &= \sum_{k=0}^{n} \binom{n}{k} z^k \\ &= 1 + nz + \frac{n(n-1)}{2!} z^2 + \cdots + z^n.\end{aligned}$$

(b) If $z_1, z_2 \in \mathbb{C}$ and $n \in \mathbb{N}$, then

$$\begin{aligned}(z_1+z_2)^n &= \sum_{k=0}^{n} \binom{n}{k} z_1^{n-k} z_2^k \\ &= z_1^n + n z_1^{n-1} z_2 + \frac{n(n-1)}{2!} z_1^{n-2} z_2^2 + \cdots + z_2^n.\end{aligned}$$

Remarks

1. Exercise 1.5(a) is the special case of part (b) of the Binomial Theorem with $n = 3$.
2. You may have noticed that the formula

$$(1+z)^n = 1 + nz + \frac{n(n-1)}{2!} z^2 + \cdots + z^n \tag{1.2}$$

from part (a) of the Binomial Theorem is a little misleading if $n = 1$ or $n = 2$. For example, if $n = 1$, then there are only two terms in the expansion of $(1+z)^1$, not four (or more). In this module, we adopt the convention that formulas presented in the manner of equation (1.2) are

valid for all sufficiently large values of n for which they make sense (typically $n > 2$ should suffice), and for small values of n the formula should be interpreted in the sensible way by omitting some terms.

3. It is worth remembering that the coefficients of powers of z that appear in the Binomial Theorem can be arranged in the form of Pascal's triangle. As you may recall, each entry in Pascal's triangle (aside from the 1s that form the sides of the triangle) is the sum of the entries above left and above right. The first few rows of the triangle are shown below.

$$\begin{array}{lccccccccc}
(1+z)^0 & & & & & 1 & & & & \\
(1+z)^1 & & & & 1 & & 1 & & & \\
(1+z)^2 & & & 1 & & 2 & & 1 & & \\
(1+z)^3 & & 1 & & 3 & & 3 & & 1 & \\
(1+z)^4 & 1 & & 4 & & 6 & & 4 & & 1 \\
\vdots & & & & & \vdots & & & &
\end{array}$$

Exercise 1.6

Use the Binomial Theorem to simplify the following expressions.

(a) $(1+i)^4$ (b) $(3+2i)^3$

Next we state two forms of an identity that can be used to sum a finite geometric series. The proof is the same as in the real case, so again it is omitted.

Theorem 1.3 Geometric Series Identity

(a) If $z \in \mathbb{C}$ and $n \in \mathbb{N}$, then

$$1 - z^n = (1-z)(1 + z + z^2 + \cdots + z^{n-1}).$$

(b) If $z_1, z_2 \in \mathbb{C}$ and $n \in \mathbb{N}$, then

$$z_1^n - z_2^n = (z_1 - z_2)(z_1^{n-1} + z_1^{n-2}z_2 + z_1^{n-3}z_2^2 + \cdots + z_2^{n-1}).$$

Remarks

1. Exercise 1.5(b) is the special case of part (b) of the Geometric Series Identity with $n = 3$.
2. On replacing n with $n+1$, the first of these two identities can be written as

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}, \quad \text{for } z \neq 1.$$

This is the familiar formula for summing a finite geometric series.

Exercise 1.7

(a) Use the Geometric Series Identity to simplify the expression

$$1 + (1+i) + (1+i)^2 + (1+i)^3.$$

(b) Use the Geometric Series Identity to find a factor of

$$z^5 - i$$

of the form $z - a$, for some complex number a.

(*Hint*: $i^5 = i$.)

Further exercises

Exercise 1.8

Complete the following table.

z	$\operatorname{Re} z$	$\operatorname{Im} z$	$-z$	$\overline{z}$
$2+3i$				
$-3-i$				
$4i$				
5				
0				

Exercise 1.9

Express each of the following complex numbers in Cartesian form.

(a) i^3 (b) i^4 (c) $(1+i)^2$ (d) $(1-i)^2$ (e) $\dfrac{1}{1-i}$

(f) $\dfrac{1+i}{1-i}$ (g) $(1+i)^3$ (h) $(2+i)^2 - (2-i)^2$ (i) $\dfrac{3+5i}{2-3i}$

(j) $\dfrac{3+2i}{1+4i}$ (k) $(3+4i)^4 - (3-4i)^4$ (l) $1 + i + i^2 + \cdots + i^{10}$

(m) $1 - i + i^2 - \cdots + i^{10}$

Exercise 1.10

Write down the real part, imaginary part and complex conjugate of each of the complex numbers in parts (a), (e), (g) of Exercise 1.9.

Exercise 1.11

Prove that $\operatorname{Im} \overline{z} = -\operatorname{Im} z$.

2 The complex plane

After working through this section, you should be able to:

- determine the *modulus* of a given complex number
- determine the *principal argument* and other *arguments* of a given non-zero complex number
- convert a complex number in Cartesian form to *polar form*, and vice versa
- interpret geometrically the sum, product and quotient of two complex numbers
- state De Moivre's Theorem, and use it to evaluate powers of complex numbers.

2.1 Cartesian coordinates

In this section we describe a geometric interpretation of complex numbers, and we see how this interpretation gives us insight into the properties of complex numbers.

You are probably familiar with the idea of representing an ordered pair (x, y) from $\mathbb{R}^2$ by a point on a plane, called a Cartesian plane, with horizontal coordinate x and vertical coordinate y (the Cartesian coordinates of the point). It is common to refer to 'the point (x, y)'. Complex numbers can likewise be represented by points on a Cartesian plane – the complex number $z = x + iy$ is represented by the point (x, y).

For example, the number $4 + 3i$ is represented by the point $(4, 3)$, and in Figure 2.1 this point is labelled $4 + 3i$.

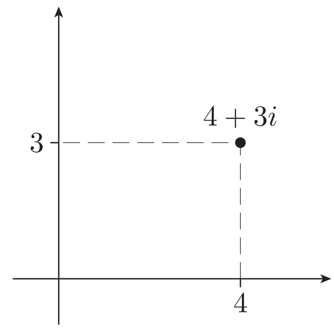


Figure 2.1 The point $4 + 3i$ in the complex plane

Thus we often speak of 'the point $z = x + iy$' and, with this interpretation, refer to the plane as the *complex plane*.

Definitions

The **complex plane** or ***z*-plane** is a Cartesian plane used to represent the set of all complex numbers in which the complex number $z = x + iy$ is represented by the point (x, y).

The horizontal axis of the complex plane is called the **real axis** and the vertical axis is called the **imaginary axis**.

Since the complex plane represents the set of all complex numbers, we denote it by the symbol $\mathbb{C}$.

In drawing the complex plane, we do not usually label the axes x and y unless it is helpful to do so (as in Unit A2, for example).

The four infinite regions of the complex plane separated off by (and not including) the axes are called **quadrants**. We label them upper-right, upper-left, lower-left and lower-right quadrants, as shown in Figure 2.2.

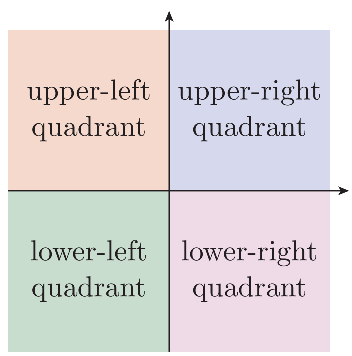


Figure 2.2 The four quadrants

The various operations on complex numbers described in Section 1 can all be given geometric interpretations in the complex plane. For example, if z is a complex number, then, as shown in Figures 2.3(a) and 2.3(b),

$-z$ is obtained by rotating z through the angle π about the origin

$\overline{z}$ is obtained by reflecting z in the real axis.

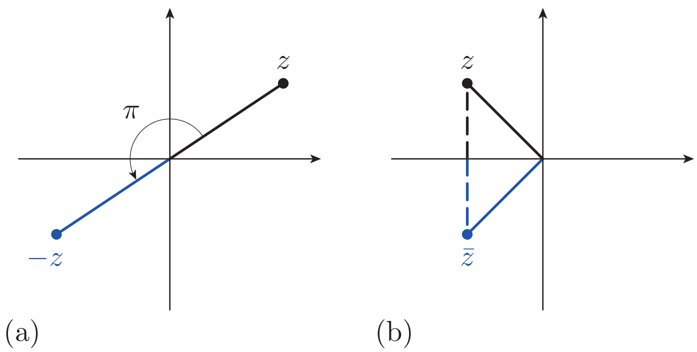


Figure 2.3 Transforming z by (a) rotating through the angle π about the origin, (b) reflecting in the real axis

Complex numbers can also be thought of as vectors, with the complex number $x + iy$ corresponding to the vector from the point $(0, 0)$ to the point (x, y). It follows that the sum of two complex numbers, and also their difference, satisfy the *parallelogram law* for vectors, as shown in Figure 2.4. In that figure, the point $z_1 - z_2$ is obtained by observing that $z_1 - z_2 = z_1 + (-z_2)$ and then applying the additive version of the parallelogram law to z_1 and $-z_2$.

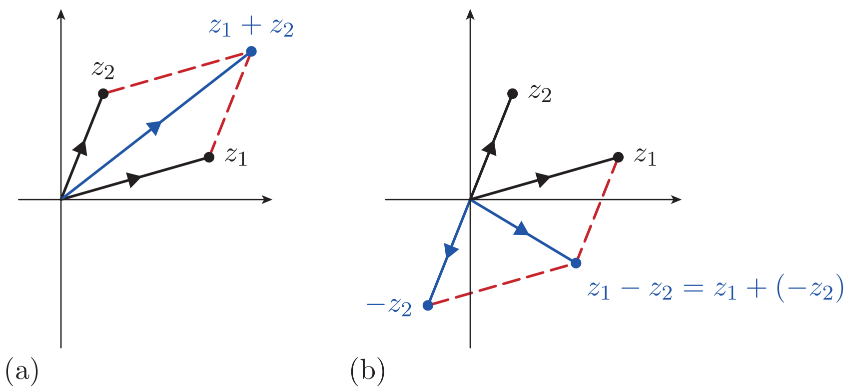


Figure 2.4 Using the parallelogram law to (a) add z_1 and z_2, (b) subtract z_2 from z_1

Exercise 2.1

With $z_1 = 3 + i$ and $z_2 = -1 + 2i$, plot the following numbers (on two separate diagrams).

(a) z_1, z_2, $-z_1$, $-z_2$, $z_1 + z_2$, $z_1 - z_2$

(b) z_1, z_2, $\overline{z_1}$, $\overline{z_2}$, $z_1 + z_2$, $\overline{z_1 + z_2}$

History of the complex plane

The complex plane is often called the **Argand diagram**, after a French mathematician with the surname Argand, who in 1806 wrote an essay on representing complex numbers by directed line segments in a plane. It is sometimes suggested that this mathematician is the Swiss-born man Jean-Robert Argand (1768–1822), but there is no evidence that this is so.

In fact, the idea of representing complex numbers in a plane had been proposed before, by the Norwegian–Danish surveyor and cartographer Caspar Wessel (1745–1818). Wessel's work, published in a little-known Danish journal in 1799, went largely unnoticed for the next century, and Wessel published no other papers in mathematics.

The value of the complex plane in representing complex numbers gained general acceptance with the work of the eminent German mathematician Carl Friedrich Gauss (1777–1855). There is evidence that he understood this geometric representation of complex numbers in his doctoral dissertation of 1799, and the idea appears explicitly in his letters to colleagues in the years to follow. Gauss was the first to consider complex numbers as points in the plane rather than just as directed line segments (as considered by Argand and Wessel). He introduced the terminology 'complex numbers' in place of 'imaginary numbers', because he thought that the old terminology was unhelpful, ascribing mystery to complex numbers and obscuring their value.

Carl Friedrich Gauss

Multiplication and division of complex numbers also have useful geometric interpretations. Before describing these, however, we need to introduce some other geometric concepts.

2.2 Polar form

The *modulus*, or *absolute value*, of a real number x is defined as

$$|x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

Equivalently, $|x|$ is the distance along the real line from 0 to x. The modulus of a complex number z is similarly defined to be the distance from 0 to z in the complex plane.

Definition

The **modulus**, or **absolute value**, of a complex number $z = x + iy$ is the distance from 0 to z; it is denoted by $|z|$. Thus

$$|z| = |x + iy| = \sqrt{x^2 + y^2}.$$

Remarks

1. In this definition we use the standard convention that if $a \geq 0$, then $\sqrt{a}$ denotes the *non-negative* square root of a (non-negative means positive or zero).
2. The plural of modulus is *moduli*.

For examples of moduli, observe that

$$|3 + 4i| = \sqrt{3^2 + 4^2} = 5,$$
$$|-3| = \sqrt{(-3)^2} = 3,$$
$$|-2i| = \sqrt{(-2)^2} = 2.$$

These moduli are shown as distances in Figure 2.5.

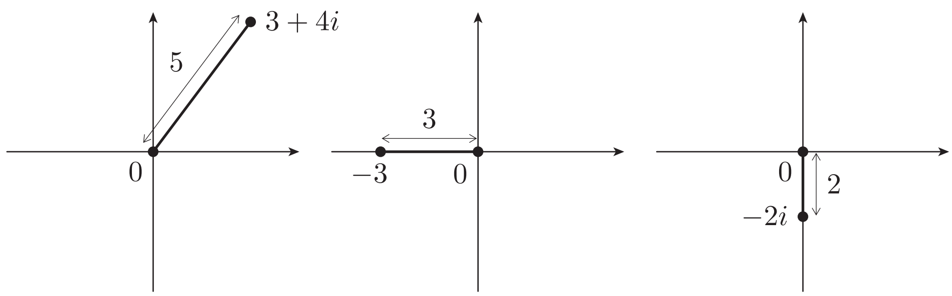


Figure 2.5 Moduli of three different complex numbers

Exercise 2.2

(a) Evaluate the following moduli.

(i) $|1 + i|$ (ii) $|2 - 4i|$ (iii) $|i|$ (iv) $|-5 + 12i|$

(b) Prove that $|\overline{z}| = |z|$ and $|-z| = |z|$.

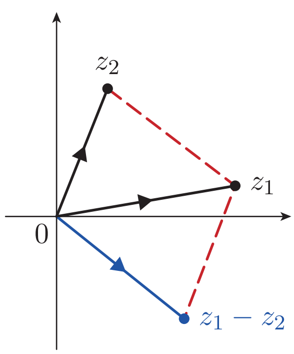


Figure 2.6 The distance from z_2 to z_1 is equal to the distance from 0 to $z_1 - z_2$

If z_1, z_2 are any two complex numbers, then, by definition, $|z_1 - z_2|$ is the distance from 0 to $z_1 - z_2$. Using the parallelogram law to add z_2 and $z_1 - z_2$ (see Figure 2.6), we deduce the following observation.

$|z_1 - z_2|$ is the distance from z_2 to z_1.

Because $z_1 + z_2 = z_1 - (-z_2)$, there is a similar geometric interpretation for $|z_1 + z_2|$.

$|z_1 + z_2|$ is the distance from $-z_2$ to z_1.

Exercise 2.3

With $z_1 = 3 + i$ and $z_2 = -1 + 2i$, determine

(a) $|z_1 - z_2|$

(b) $|z_1 + z_2|$

(c) the distance from z_2 to $-z_1$.

We now collect together various basic properties of the modulus.

Theorem 2.1 Properties of the modulus

(a) $|z| \geq 0$, with equality if and only if $z = 0$.

(b) $|\overline{z}| = |z|$ and $|-z| = |z|$.

(c) $|z|^2 = z\overline{z}$.

(d) $|z_1 - z_2| = |z_2 - z_1|$.

(e) $|z_1 z_2| = |z_1||z_2|$, and $|z_1/z_2| = |z_1|/|z_2|$ for $z_2 \neq 0$.

Property (d) says that the distance from z_2 to z_1 is the same as the distance from z_1 to z_2.

Proof Property (a) follows from the fact that $|z| = \sqrt{x^2 + y^2}$ (where $z = x + iy$), and property (b) was proved in Exercise 2.2(b). To prove property (c), note that if $z = x + iy$, then

$$z\overline{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2.$$

Property (d) follows from property (b), since $z_2 - z_1 = -(z_1 - z_2)$.

Each of the identities in property (e) can be proved by writing $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ and then calculating both sides. However, it is neater to use property (c) and Theorem 1.1, as follows:

$$\begin{aligned} |z_1 z_2|^2 &= (z_1 z_2)\overline{(z_1 z_2)} && \text{(property (c))} \\ &= (z_1 z_2)(\overline{z_1}\,\overline{z_2}) && \text{(Theorem 1.1(b)(iii))} \\ &= (z_1\overline{z_1})(z_2\overline{z_2}) && \text{(associativity and commutativity)} \\ &= |z_1|^2|z_2|^2 && \text{(property (c))}, \end{aligned}$$

so $|z_1 z_2| = |z_1||z_2|$. Similarly, if $z_2 \neq 0$ (so $\overline{z_2} \neq 0$, and $|z_2| > 0$ using property (a)), then

$$\begin{aligned} |z_1/z_2|^2 &= (z_1/z_2)\overline{(z_1/z_2)} = (z_1/z_2)(\overline{z_1}/\overline{z_2}) \\ &= (z_1\overline{z_1})/(z_2\overline{z_2}) = |z_1|^2/|z_2|^2, \end{aligned}$$

so $|z_1/z_2| = |z_1|/|z_2|$. ■

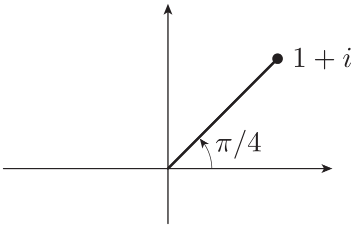


Figure 2.7 An angle for $1+i$

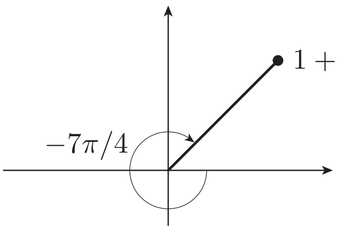


Figure 2.8 Another angle for $1+i$

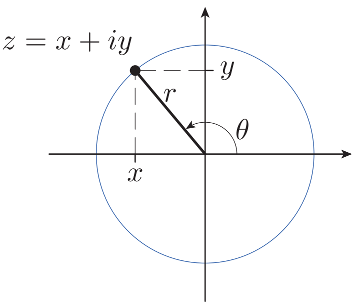


Figure 2.9 An argument of z

If the modulus $|z|$ of a complex number z is equal to 0, then z itself must equal 0 (and vice versa). However, the modulus of a *non-zero* complex number does not determine the number completely; all the points that lie on the circle of radius r centred at the origin have the same modulus, namely r. We can determine the non-zero complex number z completely by giving its modulus $|z| = r$ together with the angle θ that the line from the origin to z makes with the positive real axis.

Angles used to determine position in this way are conventionally taken to be positive when measured in an anticlockwise direction from the positive real axis, and negative when measured in a clockwise direction.

For example, $1+i$ has modulus $\sqrt{2}$, and the (positive) angle that the line from the origin to $1+i$ makes with the positive real axis is $\pi/4$ (see Figure 2.7). Of course, $\pi/4$ is not the only angle that, along with the modulus $\sqrt{2}$, specifies $1+i$; any one of the angles

$$\ldots, \quad \frac{\pi}{4} - 2\pi, \quad \frac{\pi}{4}, \quad \frac{\pi}{4} + 2\pi, \quad \frac{\pi}{4} + 4\pi, \quad \ldots$$

would do just as well – for example, the (negative) angle $\pi/4 - 2\pi = -7\pi/4$ (see Figure 2.8). This feature is reflected in the following definition using the sine and cosine functions. Figure 2.9 illustrates the definition, showing one argument of $z = x + iy$.

Definition

An **argument** of a non-zero complex number $z = x + iy$ with $|z| = r$ is an angle θ (measured in radians) such that

$$\cos\theta = \frac{x}{r} \quad \text{and} \quad \sin\theta = \frac{y}{r}.$$

Remarks

1. No argument is assigned to the number 0.
2. Each non-zero complex number has infinitely many arguments, all differing by integer multiples of 2π. For example, the arguments of $1+i$ can be written collectively as
$$\frac{\pi}{4} + 2k\pi, \quad \text{where } k \in \mathbb{Z}.$$
3. As you can see in Figures 2.7, 2.8 and 2.9, arguments are represented in figures by directed arcs: anticlockwise arcs for positive arguments and clockwise arcs for negative arguments. We use directed arcs in this way to communicate whether an angle is positive or negative. In contrast, later figures (such as Figures 2.14, 2.15 and 2.16) use *undirected* arcs; these arcs always represent positive angles – the anticlockwise arrow is omitted for convenience.
4. For some complex numbers, arguments are easily obtained by plotting the point. For example, Figure 2.10 shows that $3\pi/4$ is an argument

of $-1+i$, $\pi/2$ is an argument of i and $-\pi/4$ is an argument of $1-i$. The *calculation* of arguments is dealt with later in the section.

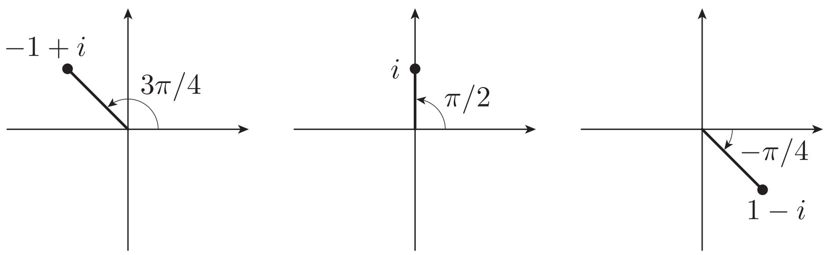


Figure 2.10 Arguments of three complex numbers

Since any non-zero complex number is completely determined by its modulus and any one of its arguments, these two quantities can be used to define an alternative coordinate system for non-zero complex numbers.

Definitions

The ordered pair (r, θ), where r is the modulus of a non-zero complex number z and θ is an argument of z, is called the **polar coordinates** of z. The expression

$$z = r(\cos\theta + i\sin\theta)$$

is said to be a representation of z in **polar form**.

Remarks

1. We will rarely use polar coordinates, preferring almost always to use polar form.
2. It follows from the definition of polar form that if $z = x + iy$, then

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

Example 2.1

Represent $-1-i$ in polar form.

Solution

Here

$$r = |-1-i| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2},$$

and, from Figure 2.11, one choice for θ is $5\pi/4$. Thus

$$-1-i = \sqrt{2}(\cos 5\pi/4 + i\sin 5\pi/4)$$

is in polar form.

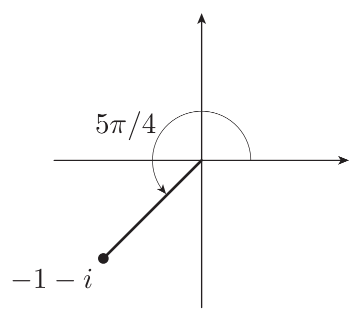


Figure 2.11 An argument of $-1-i$

Another polar form for $-1-i$ is $\sqrt{2}(\cos(-3\pi/4) + i\sin(-3\pi/4))$.

Exercise 2.4

(a) Represent the complex number i in polar form.

(b) Represent each of the following complex numbers in Cartesian form.

(i) $2(\cos \pi/3 + i \sin \pi/3)$

(ii) $3(\cos(-\pi/4) + i \sin(-\pi/4))$

The terminology 'arg z' is often used in other texts to denote an argument of a non-zero complex number z. Without further information, however, the expression $\arg z$ is ambiguous, since z has infinitely many arguments, so we will avoid using it. Instead we select one argument for special attention and call this the *principal argument* (a shortened version of the more conventional 'principal value of the argument').

Definition

The **principal argument** of a non-zero complex number z is the unique argument θ of z satisfying $-\pi < \theta \leq \pi$; it is denoted by

$$\theta = \operatorname{Arg} z.$$

(Note the capital A in Arg.)

Since the arguments of a non-zero complex number differ by multiples of 2π, *exactly one* of them satisfies $-\pi < \theta \leq \pi$.

For an example of a principal argument, the arguments of $1 + i$ are

$$\ldots, \quad -7\pi/4, \quad \pi/4, \quad 9\pi/4, \quad 17\pi/4, \quad \ldots,$$

hence $\operatorname{Arg}(1 + i) = \pi/4$ because $-\pi < \pi/4 \leq \pi$.

For complex numbers z such as $1 + i$, it is easy to determine $\operatorname{Arg} z$ by inspection. In general, the following strategy can be applied. (There are other equally valid strategies.) The figures to illustrate the strategy each have a small circle at the origin to indicate that the strategy does not apply to $z = 0$.

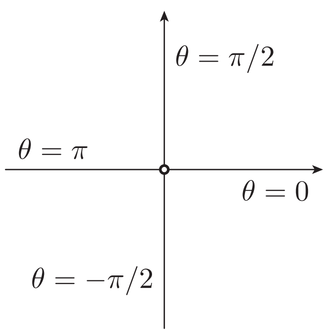


Figure 2.12 Principal arguments on the axes

Strategy for determining principal arguments

To determine the principal argument θ of a non-zero complex number $z = x + iy$, apply the relevant case below.

Case 1 If z lies on one of the axes, then θ is evident (see Figure 2.12).

Case 2 If z does not lie on one of the axes, then carry out the following two steps.

(i) Decide in which quadrant z lies (by plotting z if necessary), and then calculate the acute angle

$$\phi = \tan^{-1}(|y|/|x|)$$

in radians (see Figure 2.13(a)).

(ii) Obtain θ in terms of ϕ by using the appropriate formula in Figure 2.13(b).

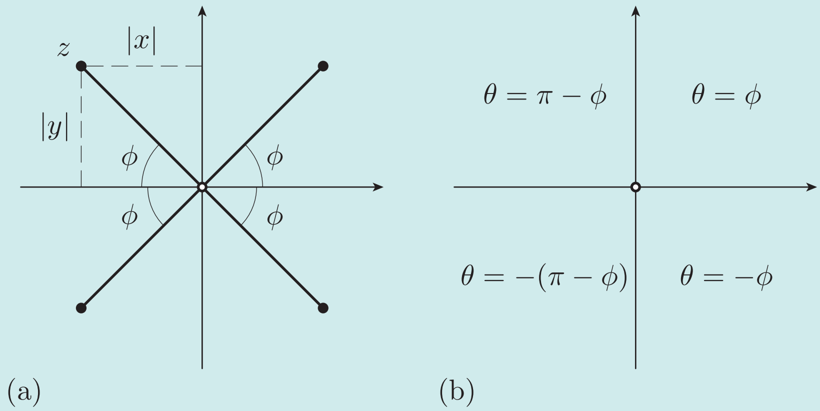


Figure 2.13 Formulas for principal arguments in the four quadrants

Remarks

1. Having found $\operatorname{Arg} z$, the other arguments of z can be obtained by adding integer multiples of 2π to $\operatorname{Arg} z$.
2. Some texts define the principal argument $\operatorname{Arg} z$ to be the argument of z that lies in the interval $[0, 2\pi)$ rather than in the interval $(-\pi, \pi]$ that we use.

Example 2.2

Find the principal argument of each of the following complex numbers.

(a) $1 + 2i$ (b) $-1 - \sqrt{3}i$ (c) $-1 + \sqrt{3}i$

Solution

We apply the strategy for determining principal arguments (case 2 each time).

(a) $1 + 2i$ lies in the upper-right quadrant (Figure 2.14), and

$$\phi = \tan^{-1}(2/1) = \tan^{-1} 2;$$

thus the principal argument θ is

$$\begin{aligned} \theta &= \phi && \text{(Figure 2.13(b))} \\ &= \tan^{-1} 2 && \text{(approximately 1.11 radians).} \end{aligned}$$

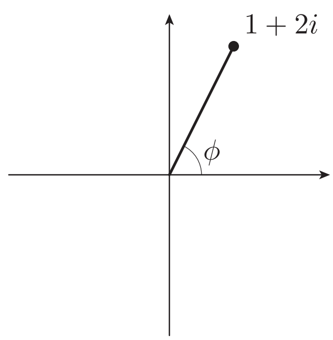


Figure 2.14 Angle ϕ for $1 + 2i$

(b) $-1-\sqrt{3}i$ lies in the lower-left quadrant (Figure 2.15), and

$$\phi = \tan^{-1}(|-\sqrt{3}|/|-1|) = \tan^{-1}\sqrt{3} = \pi/3;$$

thus the principal argument θ is

$$\begin{aligned}\theta &= -(\pi - \phi) \quad \text{(Figure 2.13(b))}\\ &= -2\pi/3.\end{aligned}$$

(c) $-1+\sqrt{3}i$ lies in the upper-left quadrant (Figure 2.16), and

$$\phi = \tan^{-1}(\sqrt{3}/|-1|) = \tan^{-1}\sqrt{3} = \pi/3;$$

thus the principal argument θ is

$$\begin{aligned}\theta &= \pi - \phi \quad \text{(Figure 2.13(b))}\\ &= 2\pi/3.\end{aligned}$$

(We will see in the next subsection that $\operatorname{Arg}\overline{z} = -\operatorname{Arg} z$, and using this observation we could deduce part (c) from part (b).)

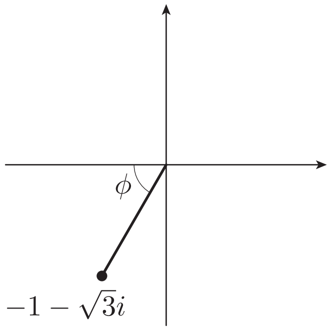


Figure 2.15 Angle ϕ for $-1-\sqrt{3}i$

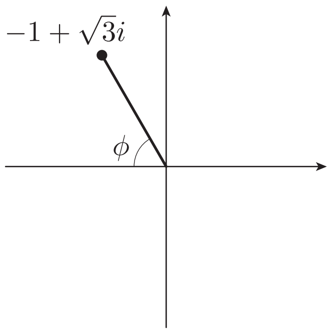


Figure 2.16 Angle ϕ for $-1+\sqrt{3}i$

Exercise 2.5

For each of the following complex numbers z, write down $\operatorname{Arg} z$ and express z in polar form.

(a) -4 (b) $3\sqrt{3}+3i$ (c) $\sqrt{3}-i$ (d) $-1-i$

We finish this subsection with a remark on equality of complex numbers in polar form. Suppose that $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ are non-zero complex numbers, and suppose also that $z_1 = z_2$. It follows that the moduli r_1 and r_2 must be equal, because $r_1 = |z_1| = |z_2| = r_2$. In contrast, the arguments θ_1 and θ_2 need not be equal – they may differ by an integer multiple of 2π; that is, $\theta_1 = \theta_2 + 2k\pi$, for some integer k. However, the *principal* arguments $\operatorname{Arg} z_1$ and $\operatorname{Arg} z_2$ must be equal, because they are uniquely specified by z_1 and z_2. Since $|z|$ and $\operatorname{Arg} z$ themselves uniquely specify the non-zero complex number z, we obtain the following conclusion.

Two non-zero complex numbers z_1 and z_2 are equal if and only if $|z_1| = |z_2|$ and $\operatorname{Arg} z_1 = \operatorname{Arg} z_2$.

2.3 A geometric interpretation of multiplication and division

A geometric interpretation of the multiplication of complex numbers can be given using the polar form of complex numbers. Indeed, if z_1 and z_2 are non-zero complex numbers with polar forms

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2),$$

then, by using the formulas for the sine and cosine of the sum of two angles, we see that

$$\begin{aligned} z_1z_2 &= r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1r_2((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)) \\ &= r_1r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)). \end{aligned}$$

So we have the following formula.

$$z_1z_2 = r_1r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)). \tag{2.1}$$

This formula shows that $|z_1z_2| = r_1r_2 = |z_1||z_2|$, which we knew already, and also that the number $\theta_1+\theta_2$ is an argument of z_1z_2. Thus we can describe in words the effect of multiplying z_1 by z_2 (both non-zero) as follows.

The *product* of the modulus of z_1 and the modulus of z_2 is the modulus of z_1z_2.

The *sum* of an argument of z_1 and an argument of z_2 is an argument of z_1z_2.

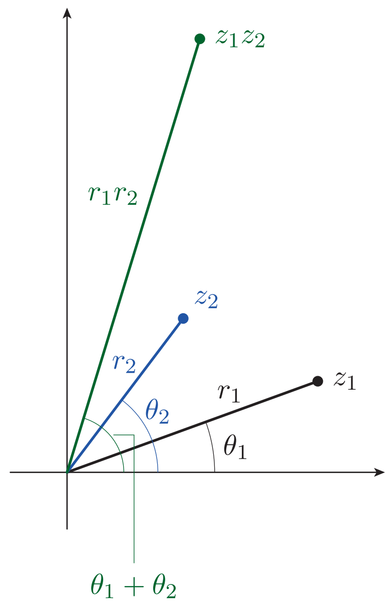


Figure 2.17 Moduli and arguments of z_1, z_2 and z_1z_2

So the geometric effect on z_1 of multiplying it by z_2 is to scale it by the factor $|z_2|$ and rotate it about 0 through the angle $\operatorname{Arg} z_2$. (This rotation is anticlockwise if $\operatorname{Arg} z_2 > 0$ and clockwise if $\operatorname{Arg} z_2 < 0$.) This is illustrated in Figure 2.17 for the case where z_1, z_2 and z_1z_2 are in the upper-right quadrant, and θ_1, θ_2 and $\theta_1+\theta_2$ are their principal arguments.

Notice that it is not always true that the principal argument of z_1z_2 is the sum of the principal arguments of z_1 and z_2. It may differ from this sum by $\pm 2\pi$.

For example, if $\operatorname{Arg} z_1 = \pi/2$ and $\operatorname{Arg} z_2 = 3\pi/4$, then

$$\operatorname{Arg} z_1 + \operatorname{Arg} z_2 = 5\pi/4.$$

Thus $5\pi/4$ is an argument of z_1z_2, but $5\pi/4 > \pi$ so it is not the principal argument of z_1z_2. In fact, since $-\pi < \operatorname{Arg}(z_1z_2) \le \pi$,

$$\operatorname{Arg}(z_1z_2) = 5\pi/4 - 2\pi = -3\pi/4.$$

Similarly, if $\operatorname{Arg} z_1 = -\pi/4$ and $\operatorname{Arg} z_2 = -7\pi/8$, then

$$\operatorname{Arg} z_1 + \operatorname{Arg} z_2 = -9\pi/8,$$

which is *an* argument of $z_1 z_2$, but

$$\operatorname{Arg}(z_1 z_2) = -9\pi/8 + 2\pi = 7\pi/8.$$

In general, since $-2\pi < \operatorname{Arg} z_1 + \operatorname{Arg} z_2 \leq 2\pi$, we have the following property of $\operatorname{Arg} z$.

If z_1 and z_2 are (non-zero) complex numbers, then

$$\operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2 + 2n\pi,$$

where n is -1, 0 or 1, depending on whether $\operatorname{Arg} z_1 + \operatorname{Arg} z_2$ is greater than π, lies in the interval $(-\pi, \pi]$, or is less than or equal to $-\pi$.

Exercise 2.6

Use polar forms of the complex numbers

$$z_1 = -1 - \sqrt{3}i \quad \text{and} \quad z_2 = 3\sqrt{3} + 3i$$

to evaluate $z_1 z_2$ and z_1^2.

(You will find Example 2.2(b) and Exercise 2.5(b) useful.)

Exercise 2.7

Describe the geometric effect on a complex number z of multiplying z by $2i$.

As you might expect, the polar form of complex numbers is also useful for division. Indeed, if z_1 and z_2 are non-zero complex numbers with polar forms

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2),$$

then, by using the formula $\cos^2\theta_2 + \sin^2\theta_2 = 1$ and the formulas for the sine and cosine of the difference of two angles, we see that

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \\
&= \frac{r_1}{r_2}\,\frac{(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)} \\
&= \frac{r_1}{r_2}\left(\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}\right) \\
&= \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)).
\end{aligned}$$

So we have the following formula.

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)). \tag{2.2}$$

This formula shows that $|z_1/z_2| = r_1/r_2 = |z_1|/|z_2|$ and also that the number $\theta_1 - \theta_2$ is an argument of z_1/z_2. Thus we can describe the effect of dividing non-zero complex numbers as follows.

The modulus of z_1 *divided* by the modulus of z_2 is the modulus of z_1/z_2.

An argument of z_1 *minus* an argument of z_2 is an argument of z_1/z_2.

Thus the geometric effect on z_1 of dividing it by z_2 is to scale it by the factor $1/|z_2|$ and rotate it about 0 through the angle $-\operatorname{Arg} z_2$. (This rotation is clockwise if $\operatorname{Arg} z_2 > 0$ and anticlockwise if $\operatorname{Arg} z_2 < 0$.)

Exercise 2.8

Use polar forms of the complex numbers

$$z_1 = 1 + \sqrt{3}i \quad \text{and} \quad z_2 = \sqrt{3} - i$$

to evaluate z_1/z_2.

Exercise 2.9

Describe the geometric effect on a complex number z of dividing z by $2i$.

An important special case of formula (2.2) for the quotient z_1/z_2 is obtained when

$$z_1 = 1 \quad \text{and} \quad z_2 = r(\cos\theta + i\sin\theta),$$

so

$$r_1 = 1, \quad \theta_1 = 0, \quad r_2 = r, \quad \theta_2 = \theta.$$

In this case we find that

$$\begin{aligned}\frac{1}{r(\cos\theta + i\sin\theta)} &= \frac{1(\cos 0 + i\sin 0)}{r(\cos\theta + i\sin\theta)}\\ &= \frac{1}{r}(\cos(0-\theta) + i\sin(0-\theta))\\ &= \frac{1}{r}(\cos(-\theta) + i\sin(-\theta)).\end{aligned}$$

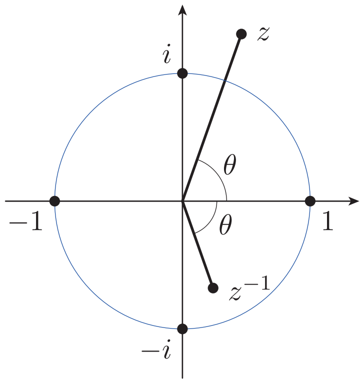


Figure 2.18 Moduli and arguments of z and z^{-1}, where $|z| > 1$

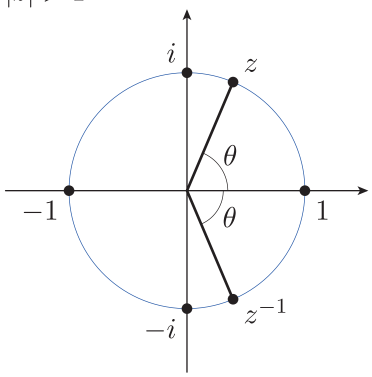


Figure 2.19 Moduli and arguments of z and z^{-1}, where $|z| = 1$

Thus the reciprocal of a non-zero complex number z can be described as follows.

> The *reciprocal* of the modulus of z is the modulus of z^{-1}.
>
> The *negative* of an argument of z is an argument of z^{-1}.

Notice that if z lies outside the circle of radius 1 centred at 0, then $|z| > 1$, so z^{-1} lies inside this circle (because $|z^{-1}| < 1$), as shown in Figure 2.18, and vice versa. If z lies on this circle, then $|z| = 1$, so z is of the form $z = \cos\theta + i\sin\theta$ and

$$\begin{aligned} z^{-1} &= (\cos\theta + i\sin\theta)^{-1} \\ &= \cos(-\theta) + i\sin(-\theta). \end{aligned}$$

Hence z^{-1} also lies on the circle (see Figure 2.19); moreover, in this case

$$\begin{aligned} z^{-1} &= \cos(-\theta) + i\sin(-\theta) \\ &= \cos\theta - i\sin\theta \\ &= \overline{z}. \end{aligned}$$

In general, for all non-zero complex numbers z,

$$z^{-1} = \frac{\overline{z}}{z\overline{z}} = \frac{1}{|z|^2}\,\overline{z},$$

so (since $1/|z|^2$ is real and positive)

$$\operatorname{Arg} z^{-1} = \operatorname{Arg}\overline{z}.$$

Also, if $-\pi < \operatorname{Arg} z < \pi$, then

$$\operatorname{Arg}\overline{z} = -\operatorname{Arg} z,$$

since $\overline{z}$ is the reflection of z in the real axis. Thus we have the following properties of $\operatorname{Arg} z$.

> If z is non-zero and $-\pi < \operatorname{Arg} z < \pi$, then
>
> $$\operatorname{Arg}\overline{z} = \operatorname{Arg} z^{-1} = -\operatorname{Arg} z.$$

Exercise 2.10

Use a polar form of $1 + i$ to evaluate $(1 + i)^{-1}$.

The product of several complex numbers $z_1, z_2, \ldots, z_n$ has an interpretation similar to the product of two complex numbers.

The *product* of the moduli of $z_1, z_2, \ldots, z_n$ is the modulus of $z_1 z_2 \cdots z_n$.

The *sum* of arguments of $z_1, z_2, \ldots, z_n$ is an argument of $z_1 z_2 \cdots z_n$.

In other words, the product of the n complex numbers

$$z_k = r_k(\cos\theta_k + i\sin\theta_k), \quad k = 1, 2, \ldots, n,$$

is given by

$$z_1 z_2 \cdots z_n = r_1 r_2 \cdots r_n(\cos(\theta_1 + \theta_2 + \cdots + \theta_n) + i\sin(\theta_1 + \theta_2 + \cdots + \theta_n)).$$

This formula can be obtained by applying the reasoning for the case $n = 2$ repeatedly (and the formula can be proved in general using the Principle of Mathematical Induction).

Exercise 2.11

Use polar forms of the complex numbers

$$z_1 = 1 + i, \quad z_2 = 1 + \sqrt{3}i, \quad z_3 = \sqrt{3} + i,$$

to evaluate $z_1 z_2 z_3$.

In the next subsection, polar form is used to calculate powers.

2.4 De Moivre's Theorem

An important special case of the formula from the end of the previous subsection for the product $z_1 z_2 \cdots z_n$ is obtained when

$$r_1 = r_2 = \cdots = r_n = 1 \quad \text{and} \quad \theta_1 = \theta_2 = \cdots = \theta_n = \theta,$$

so

$$z_1 = z_2 = \cdots = z_n = \cos\theta + i\sin\theta.$$

In this case, the product formula becomes

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \quad n = 1, 2, \ldots.$$

This identity is due to de Moivre (pronounced 'duh mwah-vr', with a short 'uh').

Abraham de Moivre

Abraham de Moivre (1667–1754) was a French mathematician who lived much of his life in England. He first discovered a rather complicated-looking version of the identity that bears his name in 1707, and later, in 1722, refined it to give the version stated here. de Moivre also made important contributions to the theory of probability, publishing an influential text on the subject in 1711.

Abraham de Moivre

Figure 2.20 shows a geometric interpretation of de Moivre's identity. The powers of $\cos\theta + i\sin\theta$ are spaced around the circle with centre 0 and radius 1, the angle between adjacent powers being θ. Each multiplication by $\cos\theta + i\sin\theta$ gives rise to a rotation through angle θ about 0.

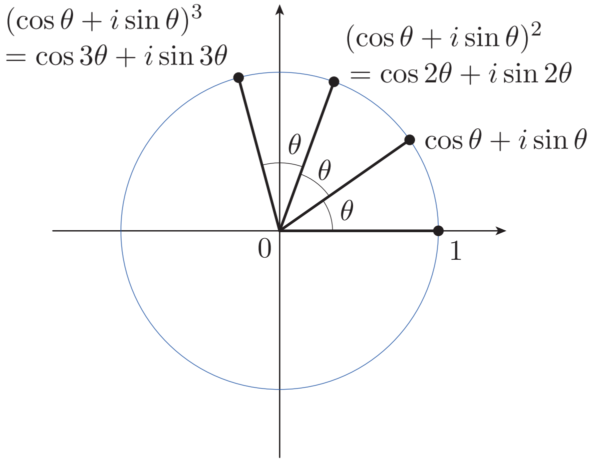


Figure 2.20 Powers of $\cos\theta + i\sin\theta$

In Figure 2.21, the position of $(\cos\theta + i\sin\theta)^{-1}$ suggests that de Moivre's identity holds also for negative integer powers; we now show that this is true.

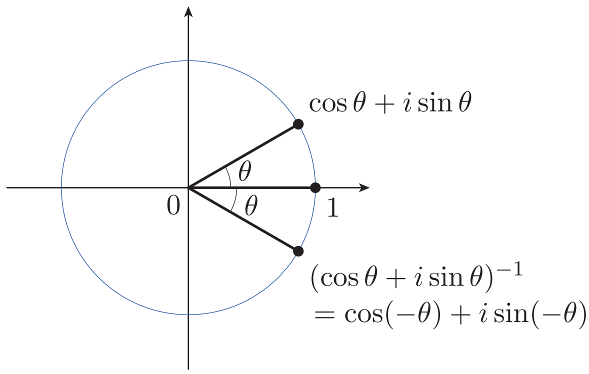


Figure 2.21 The complex number $\cos\theta + i\sin\theta$ and its reciprocal $\cos(-\theta) + i\sin(-\theta)$

Theorem 2.2 De Moivre's Theorem

If n is an integer and θ is a real number, then

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Proof We have already proved De Moivre's Theorem for a positive integer n, and it is also true for

$$n = 0, \quad \text{since } (\cos\theta + i\sin\theta)^0 = 1 = \cos 0 + i\sin 0,$$
$$n = -1, \quad \text{since } (\cos\theta + i\sin\theta)^{-1} = \cos(-\theta) + i\sin(-\theta).$$

To complete the proof, note that if m is a positive integer, then

$$\begin{aligned}(\cos\theta + i\sin\theta)^{-m} &= ((\cos\theta + i\sin\theta)^{-1})^m \\ &= (\cos(-\theta) + i\sin(-\theta))^m \\ &= \cos(-m\theta) + i\sin(-m\theta).\end{aligned}$$

Hence De Moivre's Theorem holds also if $n = -m$, where m is a positive integer. ■

Exercise 2.12

Use De Moivre's Theorem to evaluate the following powers.

(a) $(\sqrt{3} + i)^4$ (b) $(1 - \sqrt{3}i)^3$ (c) $(1 + i)^{10}$ (d) $(-1 + i)^{-8}$

(e) $(\sqrt{3} + i)^{-6}$

Further exercises

Exercise 2.13

Plot each of the following complex numbers, and express each one in polar form, using the principal argument in each case.

(a) 5 (b) i (c) $-3i$ (d) $2 + 2i$ (e) $-2 + 2i$

(f) $-\sqrt{3} - i$ (g) $3 + 4i$ (h) $3 - 4i$

Exercise 2.14

Plot each of the following complex numbers, and express each one in Cartesian form.

(a) $\cos\pi + i\sin\pi$ (b) $4(\cos(-\pi/2) + i\sin(-\pi/2))$

(c) $3(\cos 3\pi/4 + i\sin 3\pi/4)$ (d) $3(\cos\pi/6 + i\sin\pi/6)$

(e) $\cos(-2\pi/3) + i\sin(-2\pi/3)$

Exercise 2.15

Find the distance from z_1 to z_2 in each of the following cases.

(a) $z_1 = 1 + i$, $z_2 = 2 + 3i$ (b) $z_1 = -2 + 3i$, $z_2 = 1 - 7i$

(c) $z_1 = i$, $z_2 = -i$

Exercise 2.16

Use polar form and De Moivre's Theorem to evaluate the following expressions, giving your answers in Cartesian form.

(a) $(1 + \sqrt{3}i)^5$ (b) $(1 + i)^{-4}$ (c) $\dfrac{(1+i)^6}{(\sqrt{3} - i)^3}$

Exercise 2.17

Use De Moivre's Theorem and the Binomial Theorem to prove that

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta,$$

where θ is a real number.

Exercise 2.18

Prove that if $(x + iy)^4 = a + ib$, where $x + iy$ and $a + ib$ are in Cartesian form, then

$$(x^2 + y^2)^4 = a^2 + b^2.$$

Exercise 2.19

Prove that if $\overline{z} = z^{-1}$, then $|z| = 1$.

3 Solving equations with complex numbers

After working through this section, you should be able to:

- calculate the *nth roots* of a complex number
- solve certain polynomial equations with complex coefficients.

As we discussed in the Introduction, the use of complex numbers allows both quadratic and cubic equations with real coefficients to be solved. You will see in this module that complex numbers enable us to solve many equations that do not have real solutions. In this section we describe various polynomial equations whose complex solutions can be found explicitly.

3.1 Calculating nth roots

If a is a non-negative real number and n is a positive integer, then $\sqrt[n]{a}$ or $a^{1/n}$ denotes the non-negative nth root of a, that is, the unique non-negative number x such that $x^n = a$. In this subsection we discuss the nth roots of a complex number, beginning with square roots.

The simplest quadratic equation that has a complex solution but no real solutions is

$$z^2 + 1 = 0, \quad \text{that is,} \quad z^2 = -1.$$

One solution of this equation is $z = i$, since $i^2 = -1$; another solution is $z = -i$, since $(-i)^2 = (-1)^2 i^2 = -1$.

A more general quadratic equation is

$$z^2 = w,$$

where w is a given complex number. Any solution z of this equation is called a **square root** of w; for example, both i and $-i$ are square roots of -1.

In fact, we will show shortly that each non-zero complex number w has exactly two square roots. Then later we will introduce the notation $\sqrt{w}$ or $w^{1/2}$ to denote a particular square root of w.

The following example shows how to find square roots geometrically.

Example 3.1

Find the two solutions of the equation

$$z^2 = i.$$

Solution

By the geometric properties of multiplication of complex numbers, described in Subsection 2.3,

the square of the modulus of z is the modulus of z^2,

an argument of z multiplied by 2 is an argument of z^2.

Since i has modulus 1 and argument $\pi/2$, one solution of $z^2 = i$ is obtained by taking z to have modulus $\sqrt{1} = 1$ and argument $\frac{1}{2}(\pi/2) = \pi/4$ (see Figure 3.1). This gives

$$\begin{aligned} z &= 1\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \\ &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i. \end{aligned}$$

(Check: $\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^2 = \frac{1}{2} + 2\,\frac{1}{\sqrt{2}}\,\frac{1}{\sqrt{2}}i - \frac{1}{2} = i.$)

Since $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ is a solution of $z^2 = i$, and $(-1)^2 = 1$,

$$z = -\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$$

is another solution of $z^2 = i$. Therefore the required solutions are

$$z = \pm\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right),$$

illustrated in Figure 3.2.

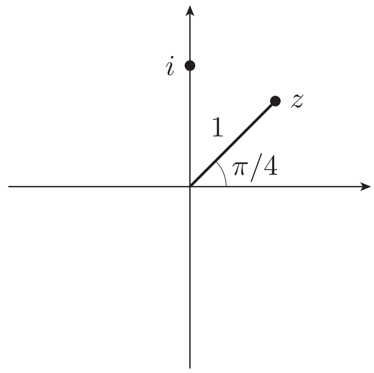


Figure 3.1 Modulus and argument of a square root of i

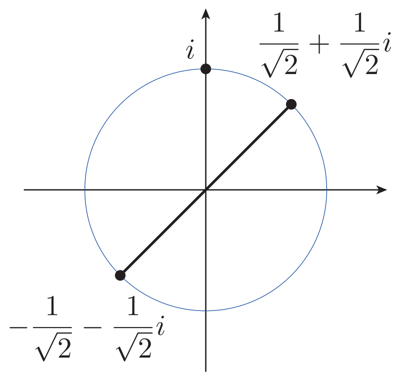


Figure 3.2 The two square roots of i

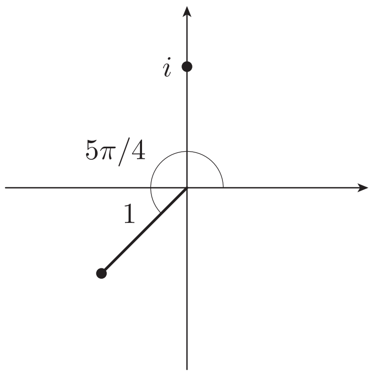


Figure 3.3 A square root of i with argument $5\pi/4$

Remarks

1. Notice that the second solution could also have been found geometrically. For example, if we had begun by taking i to have modulus 1 and argument $\pi/2 + 2\pi = 5\pi/2$, then the corresponding solution z would have modulus $\sqrt{1} = 1$ and argument $\frac{1}{2}(5\pi/2) = 5\pi/4$ (see Figure 3.3). This gives the solution
$$z = 1\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i.$$
Applying the same procedure but with other arguments for i (such as $\pi/2 - 2\pi = -3\pi/2$) just gives us repeats of the two solutions we have obtained already.
2. Note that if z is a square root of a complex number w, then $-z$ is also a square root of w.
3. An alternative method of solving $z^2 = i$ is to write $z = x + iy$, equate the real parts and imaginary parts of
$$(x+iy)^2 = x^2 - y^2 + 2xyi = i$$
(remembering that $z_1 = z_2$ means that $\operatorname{Re} z_1 = \operatorname{Re} z_2$ and $\operatorname{Im} z_1 = \operatorname{Im} z_2$), and then solve the resulting equations for x and y (as you will do in Exercise 3.7). This method is, however, not suitable for finding the nth roots of complex numbers if $n > 2$.

Exercise 3.1

Find the two solutions of the equation
$$z^2 = -1 + \sqrt{3}i.$$

We now turn to the more general equation
$$z^n = w,$$
where w is a given complex number and n is any positive integer with $n \geq 2$. Each solution of $z^n = w$ is called an ***n*th root** of w. If $w = 0$, then $z = 0$ is the only solution. We will show shortly that each non-zero complex number w has exactly n nth roots.

As a simple example, consider the equation
$$z^3 = -8.$$

We will find solutions of this equation by expressing -8 in polar form. First, we can write
$$-8 = 8(\cos\pi + i\sin\pi),$$
so a solution of $z^3 = -8$ is obtained by taking z to have modulus $\sqrt[3]{8} = 2$ and argument $\pi/3$. This gives
$$z = 2(\cos\pi/3 + i\sin\pi/3) = 1 + \sqrt{3}i.$$

But there are other ways of writing -8 in polar form, such as

$$-8 = 8(\cos 3\pi + i \sin 3\pi),$$

so another solution of $z^3 = -8$ is obtained by taking z to have modulus 2 (as before) and argument $3\pi/3 = \pi$. This is the real solution

$$z = 2(\cos \pi + i \sin \pi) = -2.$$

By writing -8 in polar form in yet another way, as

$$-8 = 8(\cos 5\pi + i \sin 5\pi),$$

we obtain a third solution of $z^3 = -8$, namely

$$z = 2(\cos 5\pi/3 + i \sin 5\pi/3) = 1 - \sqrt{3}i.$$

We have now found the three solutions

$$z = -2,\ 1 + \sqrt{3}i,\ 1 - \sqrt{3}i,$$

and there are no more; any other polar form representations of -8 will give repeats of solutions we have obtained already.

Notice that these solutions all lie on the circle with centre 0 and radius 2 (see Figure 3.4), and that the angle between adjacent solutions is $2\pi/3$. Thus these three cube roots form the vertices of an equilateral triangle. This is a special case of a general result for nth roots.

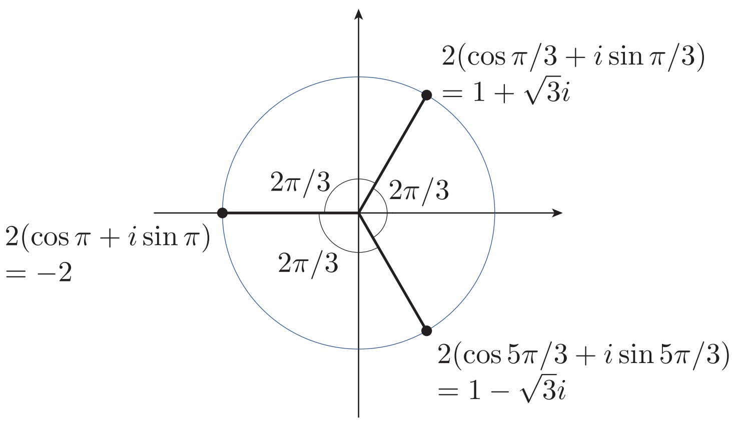


Figure 3.4 The three cube roots of -8

In the following theorem we use the Greek letters ρ and ϕ in place of r and θ, because r and θ are needed in the proof.

Theorem 3.1

Let $w = \rho(\cos \phi + i \sin \phi)$ be a non-zero complex number in polar form. Then w has exactly n nth roots, given by

$$z_k = \rho^{1/n}\left(\cos\left(\frac{\phi}{n} + k\frac{2\pi}{n}\right) + i \sin\left(\frac{\phi}{n} + k\frac{2\pi}{n}\right)\right),$$

where $k = 0, 1, \ldots, n-1$. These roots form the vertices of an n-sided regular polygon inscribed in the circle of radius $\rho^{1/n}$ centred at 0.

Proof We seek the solutions of $z^n = w$ in polar form, $z = r(\cos\theta + i\sin\theta)$. Since $w = \rho(\cos\phi + i\sin\phi)$, the equation $z^n = w$ takes the form

$$r^n(\cos\theta + i\sin\theta)^n = \rho(\cos\phi + i\sin\phi);$$

that is,

$$r^n(\cos n\theta + i\sin n\theta) = \rho(\cos\phi + i\sin\phi),$$

by De Moivre's Theorem.

We now determine r and θ by 'equating moduli and arguments'. That is, we equate the moduli of both sides, and use the fact that the arguments of the two sides differ by an integer multiple of 2π, to obtain

$$r^n = \rho \quad \text{and} \quad n\theta = \phi + 2k\pi, \quad \text{where } k \in \mathbb{Z}.$$

Since r is non-negative, the only possible value of r is $\rho^{1/n}$ (recall that for $a \geq 0$, $a^{1/n}$ means the non-negative nth root of a), and the only possible values of θ are

$$\theta = \frac{\phi}{n} + k\frac{2\pi}{n}, \quad \text{where } k \in \mathbb{Z}.$$

Hence the solutions of $z^n = w$ are all of the form

$$z_k = \rho^{1/n}\left(\cos\left(\frac{\phi}{n} + k\frac{2\pi}{n}\right) + i\sin\left(\frac{\phi}{n} + k\frac{2\pi}{n}\right)\right), \quad \text{where } k \in \mathbb{Z}.$$

At first sight, it might appear that we have found infinitely many solutions, one for each value of k. However, not all these solutions are distinct. Indeed, if k_1 and k_2 differ by an integer multiple of n, say

$$k_2 = k_1 + mn, \quad \text{where } m \in \mathbb{Z},$$

then

$$\frac{\phi}{n} + k_2\frac{2\pi}{n} = \frac{\phi}{n} + (k_1 + mn)\frac{2\pi}{n} = \left(\frac{\phi}{n} + k_1\frac{2\pi}{n}\right) + 2\pi m,$$

so

$$\frac{\phi}{n} + k_2\frac{2\pi}{n} \quad \text{and} \quad \frac{\phi}{n} + k_1\frac{2\pi}{n}$$

differ by an integer multiple of 2π.

Hence the solutions arising from k_1 and k_2 are identical. So all possible solutions of $z^n = w$ arise from the integers $k = 0, 1, \ldots, n-1$. These n solutions are clearly distinct, since they lie on the circle of radius $\rho^{1/n}$ centred at 0, with the angle $2\pi/n$ between adjacent solutions. Thus they do form the vertices of a regular n-sided polygon. (Figure 3.5 illustrates this in the case $n = 6$.) ■

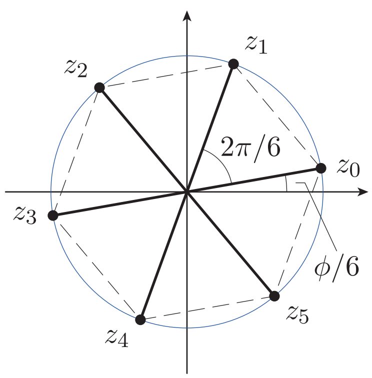


Figure 3.5 Six sixth roots

If $w = \rho(\cos\phi + i\sin\phi)$, where ϕ is the *principal argument* of w, then

$$z_0 = \rho^{1/n}\left(\cos\frac{\phi}{n} + i\sin\frac{\phi}{n}\right)$$

is called the **principal nth root** of w, denoted by $\sqrt[n]{w}$ or $w^{1/n}$. In particular, the principal square root of w is denoted by $\sqrt{w}$ or $w^{1/2}$. Note that if w is a positive real number (with principal argument $\phi = 0$), then the principal nth root of w also has argument 0, so it is positive. Hence this use of the notation $\sqrt[n]{w}$ is consistent with the familiar real case. This consistency is taken further because for $0 \in \mathbb{C}$, $\sqrt[n]{0}$ or $0^{1/n}$ is defined to be 0.

A particularly important case of Theorem 3.1 occurs when $w = 1$, so $\rho = 1$ and $\phi = 0$.

Corollary

The number 1 has exactly n nth roots, given by

$$z_k = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right), \quad k = 0, 1, \ldots, n-1.$$

These are called the ***n*th roots of unity**.

Note that $z_0 = 1$ is the principal nth root of unity for each n.

The nth roots of unity lie on the circle of radius 1 centred at 0, with the angle $2\pi/n$ between adjacent roots. The cases $n = 2, 3, 4$ are illustrated in Figure 3.6.

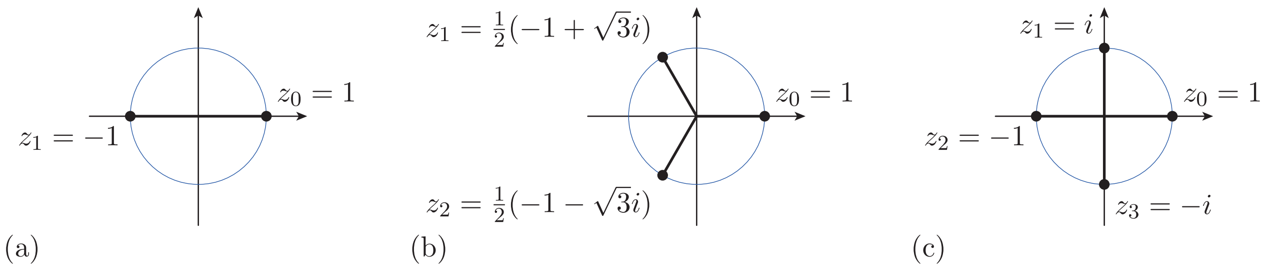


Figure 3.6 The nth roots of unity for (a) $n = 2$, (b) $n = 3$, (c) $n = 4$

Example 3.2

Determine the fourth roots of $-8 + 8\sqrt{3}i$ in polar and Cartesian forms, plot them in the complex plane, and specify the principal fourth root.

Solution

Since

$$|-8 + 8\sqrt{3}i| = |8||-1 + \sqrt{3}i| = 8\sqrt{(-1)^2 + (\sqrt{3})^2} = 16$$

and $-8 + 8\sqrt{3}i$ has principal argument

$$\pi - \tan^{-1}\frac{8\sqrt{3}}{8} = \pi - \frac{\pi}{3} = \frac{2\pi}{3},$$

we deduce that

$$-8+8\sqrt{3}i = 16\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right).$$

Hence, by Theorem 3.1 with $\rho = 16$ and $\phi = 2\pi/3$, the four fourth roots of $-8+8\sqrt{3}i$ are

$$\begin{aligned} z_k &= 16^{1/4}\left(\cos\left(\frac{2\pi/3}{4} + k\frac{2\pi}{4}\right) + i\sin\left(\frac{2\pi/3}{4} + k\frac{2\pi}{4}\right)\right) \\ &= 2\left(\cos\left(\frac{\pi}{6} + k\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{6} + k\frac{\pi}{2}\right)\right), \quad k = 0,1,2,3. \end{aligned}$$

Thus the arguments of the four fourth roots of $-8+8\sqrt{3}i$ are

$$\frac{\pi}{6}, \quad \frac{\pi}{6} + \frac{\pi}{2} = \frac{2\pi}{3}, \quad \frac{\pi}{6} + 2\left(\frac{\pi}{2}\right) = \frac{7\pi}{6}, \quad \frac{\pi}{6} + 3\left(\frac{\pi}{2}\right) = \frac{5\pi}{3},$$

so the polar and Cartesian forms of the fourth roots are as given below.

z_k	Polar form	Cartesian form
z_0	$2(\cos\pi/6 + i\sin\pi/6)$	$\sqrt{3}+i$
z_1	$2(\cos 2\pi/3 + i\sin 2\pi/3)$	$-1+\sqrt{3}i$
z_2	$2(\cos 7\pi/6 + i\sin 7\pi/6)$	$-\sqrt{3}-i$
z_3	$2(\cos 5\pi/3 + i\sin 5\pi/3)$	$1-\sqrt{3}i$

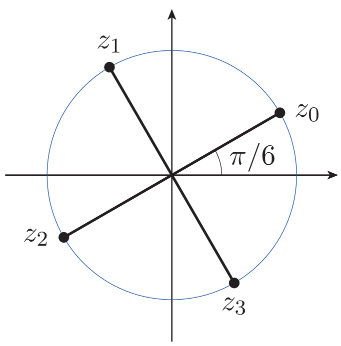


Figure 3.7 The fourth roots of $-8+8\sqrt{3}i$

The fourth roots are plotted in Figure 3.7.

Since the principal argument of $-8+8\sqrt{3}i$ is $2\pi/3$, its principal fourth root is $z_0 = \sqrt{3}+i$.

The solution above illustrates the following strategy.

Strategy for finding nth roots

To find the n nth roots $z_0, z_1, \ldots, z_{n-1}$ of a non-zero complex number w, apply the following steps.

1. Express w in polar form, with modulus ρ and argument ϕ.
2. Substitute the values of ρ and ϕ in the formula

   $$z_k = \rho^{1/n}\left(\cos\left(\frac{\phi}{n} + k\frac{2\pi}{n}\right) + i\sin\left(\frac{\phi}{n} + k\frac{2\pi}{n}\right)\right),$$

   where $k = 0, 1, \ldots, n-1$.
3. Convert the roots to Cartesian form, if required.

Remarks

1. In the first step you should normally choose ϕ to be $\operatorname{Arg} w$ (as in Example 3.2); this has the advantage that the root z_0 obtained in the second step is the principal nth root of w.

   One disadvantage of this choice is the appearance of minus signs when $\operatorname{Arg} w$ is negative. This can be avoided by choosing ϕ to be $\operatorname{Arg} w + 2\pi$ (which is positive), but with this choice, z_0 in step 2 will not be the principal nth root of w, which has to be identified separately. (See the solution to Exercise 3.2(b).)

2. Where possible you should try to use the fact that the nth roots of w form a regular n-sided polygon to *check* your calculation of nth roots. For example, in Example 3.2 note that

   $$z_1 = iz_0, \quad z_2 = i^2 z_0 = -z_0 \quad \text{and} \quad z_3 = i^3 z_0 = -iz_0,$$

   corresponding to the fact that multiplying z by i rotates z about 0 through $\pi/2$ anticlockwise.

Exercise 3.2

(a) Determine the cube roots of $8i$ in Cartesian form, plot them in the complex plane, and specify the principal cube root.

(b) Determine the sixth roots of $-i$ in polar form, plot them in the complex plane, and specify the principal sixth root.

Exercise 3.3

(a) Use the Geometric Series Identity to prove that if z is an nth root of unity ($n \geq 2$) and $z \neq 1$, then

$$1 + z + z^2 + \cdots + z^{n-1} = 0.$$

(b) Deduce from part (a) that the n nth roots of unity have sum 0.

The result of Exercise 3.3 has the following physical interpretation. Consider n identical point masses distributed evenly around a circle in a plane, at positions marked by the nth roots of unity. Then the centre of mass of this collection of point masses is at the origin.

3.2 Solutions of polynomial equations

The quadratic equation

$$az^2 + bz + c = 0,$$

where a, b, c are complex numbers and $a \neq 0$, can be solved by the methods for solving real quadratic equations. For example, we may be able to factorise the quadratic expression, as in the following cases:

$$z^2 + 9 = (z - 3i)(z + 3i) = 0, \quad \text{so } z = \pm 3i,$$

and

$$z^2 + (1-i)z - i = (z+1)(z-i) = 0, \quad \text{so } z = -1, i.$$

If there is no easy factorisation, then the formula

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{3.1}$$

can be used. The justification of this formula (by completing the square and rearranging) is identical to that in the real case.

Exercise 3.4

Solve the following equations.

(a) $z^2 - 7iz + 8 = 0$ (b) $z^2 + 2z + 1 - i = 0$

In the previous subsection we saw how to find the n solutions of the equation

$$z^n - w = 0.$$

However, it is only in exceptional cases that we can find an explicit algebraic solution of the polynomial equation (of degree $n \geq 3$)

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0,$$

where $a_0, a_1, \ldots, a_n$ are complex numbers and $a_n \neq 0$. For example, it may be possible to reduce a given polynomial equation to a quadratic equation by making a substitution, as in the next example.

Example 3.3

Solve the equation

$$z^4 + 4z^2 + 8 = 0.$$

Solution

Substituting $w = z^2$ gives

$$w^2 + 4w + 8 = 0,$$

which has solutions

$$w = \frac{-4 \pm \sqrt{16 - 32}}{2} = -2 \pm 2i.$$

Thus $z = \pm\sqrt{-2+2i}$ or $z = \pm\sqrt{-2-2i}$.

Since

$$-2 + 2i = \sqrt{8}(\cos 3\pi/4 + i \sin 3\pi/4),$$

we have

$$\sqrt{-2+2i} = 8^{1/4}(\cos 3\pi/8 + i \sin 3\pi/8).$$

(This is the principal square root of $-2+2i$, because $3\pi/4$ is the principal argument of $-2+2i$.) Thus two solutions of $z^4+4z^2+8=0$ are $\pm 8^{1/4}(\cos 3\pi/8 + i\sin 3\pi/8)$.

Similarly, since

$$-2-2i = \sqrt{8}(\cos(-3\pi/4) + i\sin(-3\pi/4)),$$

we have

$$\sqrt{-2-2i} = 8^{1/4}(\cos(-3\pi/8) + i\sin(-3\pi/8));$$

thus two further solutions are $\pm 8^{1/4}(\cos(-3\pi/8) + i\sin(-3\pi/8))$.

So the four solutions are

$$\pm 8^{1/4}(\cos 3\pi/8 + i\sin 3\pi/8)$$

and

$$\pm 8^{1/4}(\cos(-3\pi/8) + i\sin(-3\pi/8)).$$

Remark

The solutions in Example 3.3 are presented using the $\pm$ notation to indicate the relationship between them. (As a consequence, they are not all in polar form.) In fact, since $\cos(-3\pi/8) = \cos 3\pi/8$ and $\sin(-3\pi/8) = -\sin 3\pi/8$, the four solutions form two complex conjugate pairs. It can be shown that non-real roots of a polynomial equation with real coefficients must occur in complex conjugate pairs; you will see this in Exercise 3.9.

Exercise 3.5

(a) Solve the equation

$$z^6 - 7iz^3 + 8 = 0.$$

(*Hint*: Use Exercise 3.4(a) and then Exercise 3.2(a). Also, you may find the following fact useful: if z is a cube root of $8i$, then $-\frac{1}{2}z$ is a cube root of $-i$.)

(b) Solve the equation

$$z^4 + 4iz^2 + 8 = 0.$$

Further exercises

Exercise 3.6

For each of the following complex numbers determine, in Cartesian form where convenient, the nth roots indicated, and plot them. In each case specify the principal nth root.

(a) The square roots of

(i) $-i$ (ii) $4i$.

(b) The cube roots of

(i) -1 (ii) $-2+2i$.

(c) The fourth roots of

(i) $\dfrac{1}{\sqrt{2}}(-1-i)$ (ii) $-1+i$.

(d) The fifth roots of

(i) -1 (ii) $-16+16\sqrt{3}i$.

Exercise 3.7

Use the method of equating real parts and imaginary parts to solve each of the following equations.

(a) $(x+iy)^2 = 3+4i$ (b) $(x+iy)^2 = -5+12i$

Exercise 3.8

Solve each of the following equations, and plot their solutions.

(a) $z^4 - z^2 + 1 + i = 0$ (b) $z^3 - 4z^2 + 6z - 4 = 0$

Exercise 3.9

Let $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$, where $a_0, a_1, \ldots, a_n$ are real numbers. Prove that if z satisfies $p(z) = 0$, then $p(\overline{z}) = 0$.

(This shows that non-real roots of a polynomial equation with real coefficients must occur in complex conjugate pairs.)

4 Sets of complex numbers

After working through this section, you should be able to:

- understand the meaning of an inequality between real expressions involving complex numbers
- understand the specification of subsets of the complex plane in terms of such inequalities
- recognise certain basic *open* and *closed sets*.

4.1 Inequalities

Throughout the module we will use many inequalities involving complex numbers, and you will need to become adept at interpreting them. Here are some simple inequalities involving a complex number z and some examples of values of z for which they are true (✓) or false (×).

	$1+i$	$2-i$	$-\frac{1}{2}+\frac{1}{2}i$	$-1-3i$
$\operatorname{Re} z > 1$	×	✓	×	×
$\lvert z\rvert \leq 1$	×	×	✓	×
$\lvert\operatorname{Im} z\rvert > 2$	×	×	×	✓
$\operatorname{Arg} z < \pi/2$	✓	✓	×	✓

Notice that these four inequalities are all between expressions that are *real-valued*. We never write inequalities between complex-valued expressions such as $2+i$ or z^2+1.

The inequalities

$$z_1 < z_2 \quad \text{and} \quad z_1 \leq z_2$$

have no meaning unless both z_1 and z_2 are real.

The reason why we can use inequalities with real numbers but not with complex numbers is because $\mathbb{R}$ is an *ordered* field, but $\mathbb{C}$ is not.

Exercise 4.1

Complete the following true/false table.

	$1+2i$	$-1-2i$	i	-2
$\operatorname{Re} z < 0$				
$\lvert z\rvert > 2$				
$\operatorname{Im} z \leq -1$				
$\operatorname{Arg} z \geq 0$				

$\mathbb{C}$ is not an ordered field

Roughly speaking, an ordered field is a field whose elements can be ordered using inequalities which satisfy certain rules of the type that you will meet in Section 5. Here is a quick and informal explanation of why $\mathbb{C}$ is not an ordered field.

If $\mathbb{C}$ *were* an ordered field, then we would have either $i > 0$ or $i < 0$. In the first case, after multiplying both sides by the 'positive' number i, we obtain $i^2 > 0$, or $-1 > 0$, which is a contradiction! There is a similar argument for the second case.

4.2 Sketching subsets of the complex plane

In this subsection we will use set notation to describe various subsets of the complex plane. Before we do so, it will be helpful to recall some conventions from *real* analysis for representing intervals. For example, the *open interval* with endpoints $1, 3$ is

$$(1,3) = \{x : 1 < x < 3\},$$

and the *closed interval* with endpoints $-2, 2$ is

$$[-2,2] = \{x : -2 \leq x \leq 2\}.$$

(Here x is a real variable.) Intervals such as

$$(-\pi,\pi] = \{x : -\pi < x \leq \pi\} \quad \text{or} \quad [-\pi,\pi) = \{x : -\pi \leq x < \pi\}$$

are called *half-open* (or *half-closed*), and it is often convenient to use unbounded open and closed intervals, such as

$$(0,\infty) = \{x : x > 0\} \quad \text{(open)}$$

and

$$[1,\infty) = \{x : x \geq 1\} \quad \text{(closed)}.$$

Let us now proceed with sketching in the complex plane. We begin with the example of two straight lines, illustrated in Figure 4.1.

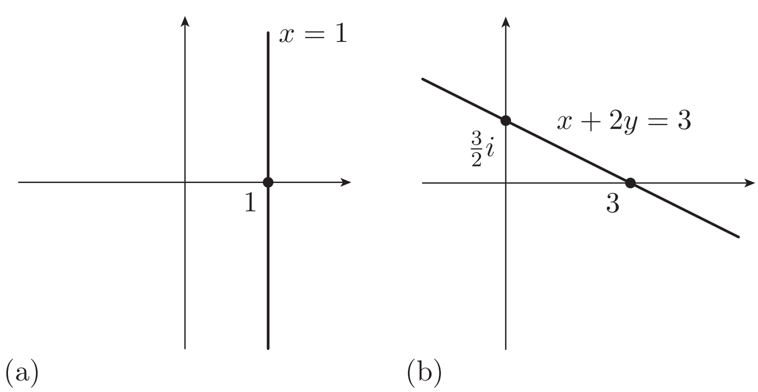


Figure 4.1 Two straight lines: (a) $\{z : \operatorname{Re} z = 1\}$, (b) $\{z : \operatorname{Re} z + 2\operatorname{Im} z = 3\}$

Figure 4.1(a) depicts the vertical straight line with equation $x = 1$. If we were working with Cartesian coordinates (x, y), then that line would be made up of all points (x, y) with $x = 1$. However, we are working in the complex plane, so the line consists of the set of complex numbers $z = x + iy$ such that $x = 1$. This set can be described in set notation as

$$\{z = x + iy : x = 1\}.$$

We can use the formula $x = \operatorname{Re} z$ to write this set without x and y as

$$\{z : \operatorname{Re} z = 1\}.$$

Similarly, the line in Figure 4.1(b) can be described as the set

$$\{z = x + iy : x + 2y = 3\}.$$

Using the formulas $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$, we can write this set as

$$\{z : \operatorname{Re} z + 2 \operatorname{Im} z = 3\}.$$

Consider now the sets represented by shading in Figure 4.2. Each set comprises all points lying to one side of a straight line (possibly including the line itself). Such sets are called **half-planes**.

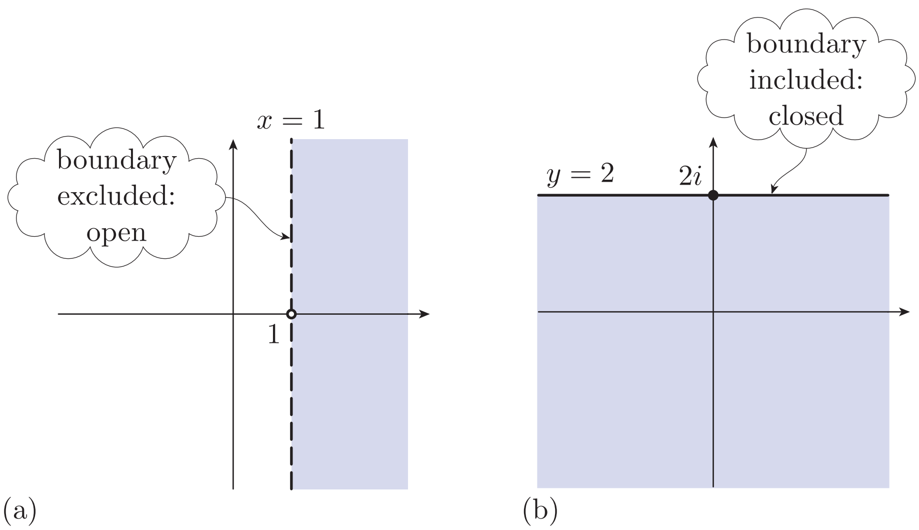


Figure 4.2 (a) Open half-plane $\{z : \operatorname{Re} z > 1\}$ (b) Closed half-plane $\{z : \operatorname{Im} z \leq 2\}$

The half-plane in Figure 4.2(a) consists of all points that lie to the right of the line $x = 1$, *excluding* the line itself. In set notation, this set is

$$\{z : \operatorname{Re} z > 1\}.$$

The boundary line $x = 1$ is drawn as a broken line, to indicate that it is excluded from the set. The line crosses the x-axis at the point 1. This point is represented by a hollow dot (a small, empty circle), to indicate that it too is excluded from the set.

The half-plane in Figure 4.2(b) is made up of all points that lie below the line $y = 2$, *including* the line itself. This set is

$$\{z : \operatorname{Im} z \leq 2\}.$$

This time, the boundary line $y = 2$ is drawn as an unbroken line, to show that it is included in the set. Also, the point $2i$ on the boundary is shown as a solid dot (a small, filled-in circle), to indicate that it is included in the set.

Since the boundary is not included in the half-plane $\{z : \operatorname{Re} z > 1\}$, we describe this half-plane as an *open* half-plane, in the same way that we describe an interval of the real line as an open interval if it does not include its boundary within the real line. (The boundary of an interval within the real line consists of its endpoints, if it has any.) In contrast, we say that the half-plane $\{z : \operatorname{Im} z \leq 2\}$ is a *closed* half-plane because it does include its boundary – again, this terminology corresponds to the terminology we use for closed intervals.

Two more examples of open and closed half-planes are shown in Figure 4.3.

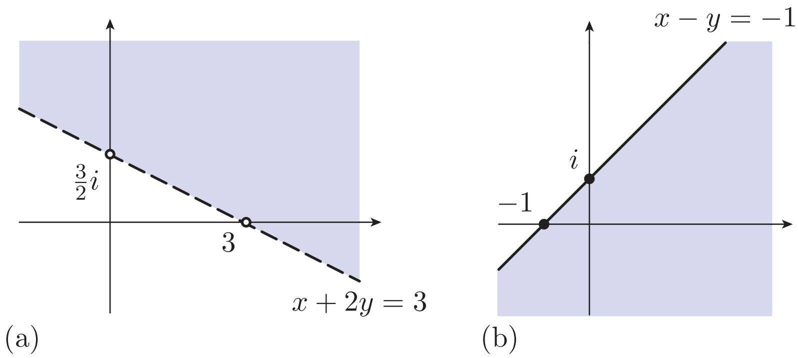


Figure 4.3 (a) Open half-plane $\{z : \operatorname{Re} z + 2\operatorname{Im} z > 3\}$
(b) Closed half-plane $\{z : \operatorname{Re} z - \operatorname{Im} z \geq -1\}$

The equation of the broken line in Figure 4.3(a) is $x + 2y = 3$, or

$$\operatorname{Re} z + 2\operatorname{Im} z = 3.$$

The shaded region represents points z that lie above this line, excluding the line itself; such points satisfy

$$\operatorname{Re} z + 2\operatorname{Im} z > 3.$$

Points below the broken line satisfy

$$\operatorname{Re} z + 2\operatorname{Im} z < 3.$$

For example, the point $z = 0$ lies below the line because

$$\operatorname{Re} 0 + 2\operatorname{Im} 0 = 0 < 3.$$

Figure 4.3(b) displays the half-plane consisting of all points that lie below the line $x - y = -1$, including the line itself. After writing this line as

$$\operatorname{Re} z - \operatorname{Im} z = -1,$$

we see that points in the half-plane satisfy

$$\operatorname{Re} z - \operatorname{Im} z \geq -1,$$

and points outside the half-plane satisfy

$$\operatorname{Re} z - \operatorname{Im} z < -1.$$

For example, the point $z = 0$ lies in the half-plane because

$$\operatorname{Re} 0 - \operatorname{Im} 0 = 0 \geq -1.$$

The half-plane in Figure 4.3(a) is an open half-plane, because the boundary is excluded, whereas the half-plane in Figure 4.3(b) is a closed half-plane, because the boundary is included. The general definitions of open and closed half-planes are as follows.

Definitions

An **open half-plane** is a set of the form

$$\{z : a \operatorname{Re} z + b \operatorname{Im} z > c\},$$

and a **closed half-plane** is a set of the form

$$\{z : a \operatorname{Re} z + b \operatorname{Im} z \geq c\},$$

where $a, b, c \in \mathbb{R}$ and a, b are not both zero.

Notice that an open half-plane can also be written in the form

$$\{z : a \operatorname{Re} z + b \operatorname{Im} z < c\}$$

(which is equally valid), by replacing a, b and c with their negatives. For example, by multiplying both sides of the inequality $\operatorname{Re} z > 1$ by -1, we can write it in the alternative form $-\operatorname{Re} z < -1$. It follows that the open half-plane shown in Figure 4.2(a) can also be written as

$$\{z : -\operatorname{Re} z < -1\}.$$

Similar comments apply to closed half-planes.

When asked to sketch a half-plane, you should first plot the boundary line $ax + by = c$, using either a broken line for an open half-plane (given by a strict inequality, $<$ or $>$) or an unbroken line for a closed half-plane (given by a weak inequality, $\leq$ or $\geq$). Then shade in one of the half-planes separated by the line. To determine which half-plane to shade, choose just one point not on the line (for example, 0) and work out whether or not it lies in the set.

Try this procedure in the following exercise.

Exercise 4.2

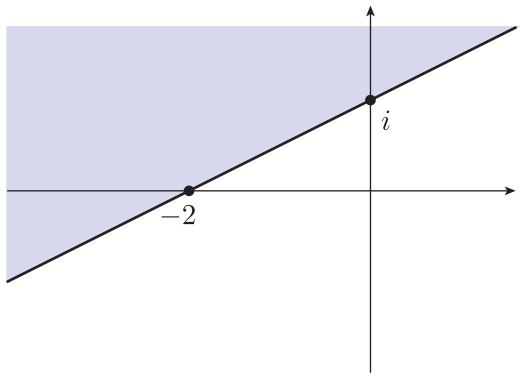
(a) Sketch the following sets.

(i) $\{z : 2\,\mathrm{Re}\,z - 3\,\mathrm{Im}\,z = -1\}$

(ii) $\{z : \mathrm{Re}\,z - \mathrm{Im}\,z > 0\}$

(iii) $\{z : \mathrm{Re}\,z + \mathrm{Im}\,z \leq -1\}$

(b) Use set notation to describe the set shaded in the following figure.



The four open half-planes shown in Figure 4.4 are particularly important in complex analysis. The half-planes above and below the real axis are called the **upper half-plane** and **lower half-plane**, respectively, and the half-planes to the left and right of the imaginary axis are called the **left half-plane** and **right half-plane**, respectively.

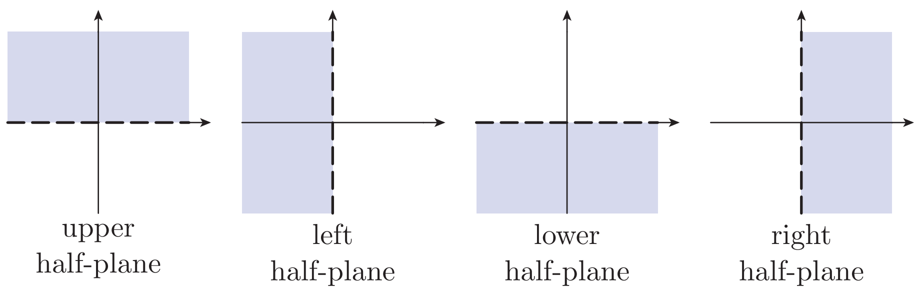


Figure 4.4 Four half-planes

Complex numbers can be used to give particularly elegant formulas for circles in the complex plane. Consider the circles shown in Figure 4.5.

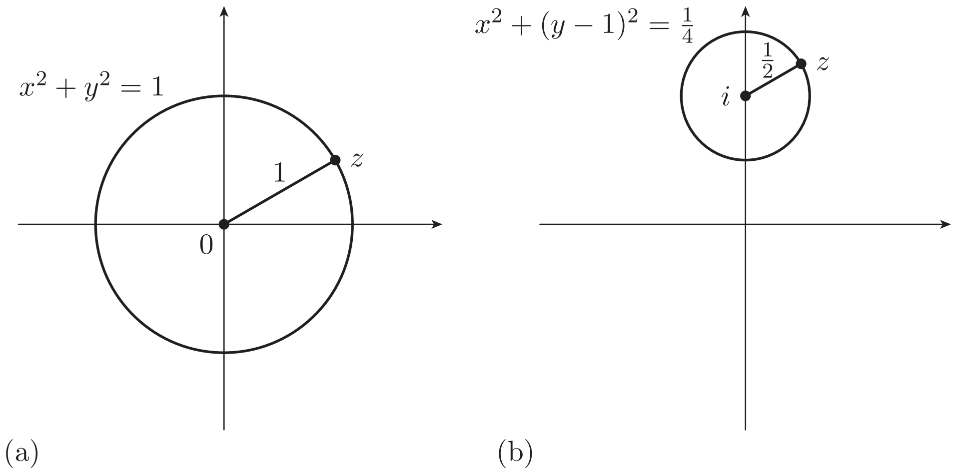


Figure 4.5 Two circles: (a) $\{z : |z| = 1\}$, (b) $\{z : |z - i| = \frac{1}{2}\}$

The circle $x^2 + y^2 = 1$ in Figure 4.5(a) is centred at 0 and has radius 1. Writing $z = x + iy$, we know that the modulus $|z|$ of z satisfies $|z|^2 = x^2 + y^2$. Therefore the equation of the circle is $|z|^2 = 1$, or, more simply, $|z| = 1$. So the circle is given in set notation as

$$\{z : |z| = 1\}.$$

This particular circle is called the **unit circle**, and we will use it often.

We can obtain the equation $|z| = 1$ in another manner by thinking geometrically. The circle consists of those points z in the complex plane that lie at a distance 1 from 0. As the distance from 0 to z is $|z - 0| = |z|$, we again see that the equation of the circle is $|z| = 1$.

Let us apply this geometric method to obtain the equation of the circle with centre i and radius $\frac{1}{2}$ shown in Figure 4.5(b). This circle is made up of those points z that lie at a distance $\frac{1}{2}$ from i. The distance from i to z is $|z - i|$, so the equation of the circle is $|z - i| = \frac{1}{2}$. Therefore this circle is given in set notation as

$$\left\{z : |z - i| = \tfrac{1}{2}\right\}.$$

In this case, the equation of the circle in complex form, namely

$$|z - i| = \tfrac{1}{2},$$

is more concise than the equation

$$x^2 + (y-1)^2 = \tfrac{1}{4}$$

using x and y.

More generally, we have the following observation, in which we use the Greek letter α to denote the centre of a circle (a complex number).

The circle with centre $\alpha \in \mathbb{C}$ and radius $r > 0$ can be written as

$$\{z : |z - \alpha| = r\}.$$

Next we look at how to use complex numbers to represent *discs*. A **disc** is the set of points inside a circle, possibly including the circle itself. Figure 4.6 shows two discs, both centred at 0.

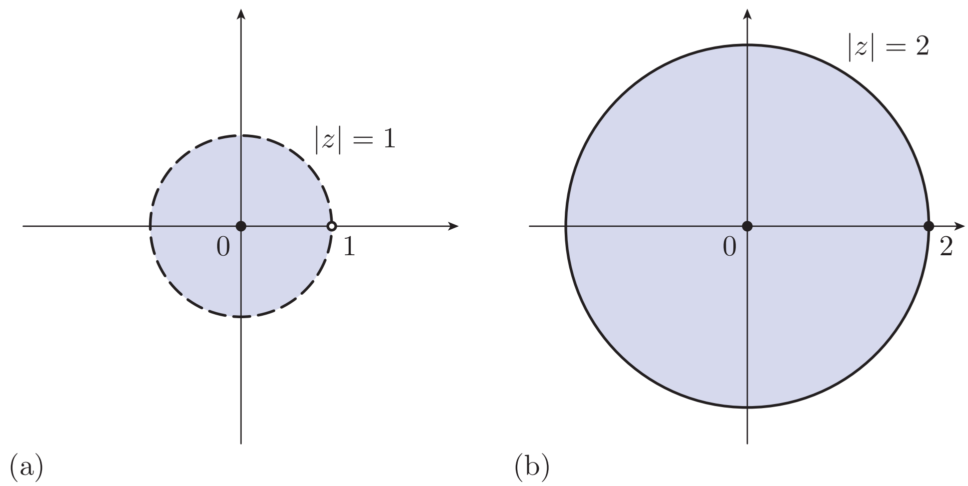


Figure 4.6 (a) Open disc $\{z : |z| < 1\}$ (b) Closed disc $\{z : |z| \leq 2\}$

The boundary circle of the disc in Figure 4.6(a) is drawn as a broken curve to indicate that it is not part of the set. It follows that this disc comprises those points that lie less than a distance 1 away from 0, so it has equation $|z| < 1$. Discs that exclude their boundaries are called *open discs*.

The boundary of the disc in Figure 4.6(b) is drawn as an unbroken curve to show that it is included in the set. This disc consists of points z that lie a distance less than or equal to 2 away from 0, so it has equation $|z| \leq 2$. Discs that include their boundaries are called *closed discs*.

Two more examples of discs are shown in Figure 4.7.

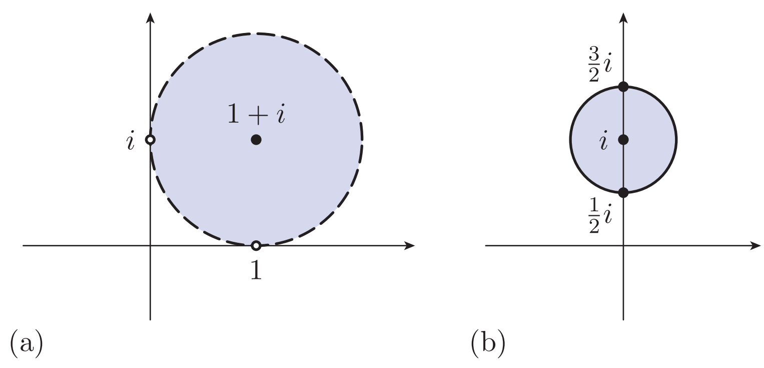


Figure 4.7 (a) Open disc $\{z : |z - 1 - i| < 1\}$ (b) Closed disc $\{z : |z - i| \leq \frac{1}{2}\}$

The disc in Figure 4.7(a) is centred at $1 + i$ and has radius 1. It is an open disc because the boundary circle, shown as a broken curve, is excluded from the set. In particular, the points $z = 1$ and $z = i$ where the circle touches the axes are not in the set. The disc is made up of those points z

that lie less than a distance 1 away from the centre $1+i$, so it has equation

$$|z-(1+i)|<1, \quad \text{or} \quad |z-1-i|<1.$$

The disc in Figure 4.7(b) with centre i and radius $\frac{1}{2}$ is a closed disc, because the boundary circle, shown as an unbroken curve, is included in the set. For example, the points $\frac{1}{2}i$ and $\frac{3}{2}i$ where the circle intersects the imaginary axis are both included in the set. This disc has equation

$$|z-i| \leq \tfrac{1}{2}.$$

More generally, we have the following definitions of open and closed discs using set notation.

Definitions

An **open disc** is a set of the form

$$\{z : |z-\alpha| < r\},$$

and a **closed disc** is a set of the form

$$\{z : |z-\alpha| \leq r\},$$

where $\alpha \in \mathbb{C}$ is the centre of the disc and $r>0$ is the radius.

Exercise 4.3

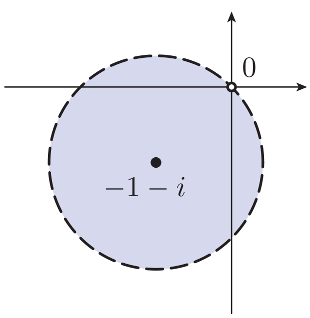
(a) Sketch the following sets.

(i) $\{z : |z-1+2i| = 1\}$

(ii) $\{z : |z-1+2i| < 1\}$

(iii) $\{z : |z+2-3i| \leq 3\}$

(b) Use set notation to describe the set shaded in the following figure, which is a disc with centre $-1-i$.



Next we will look at various other sets that are related to circles and discs. Figure 4.8 shows two shaded sets, each the outside of a disc.

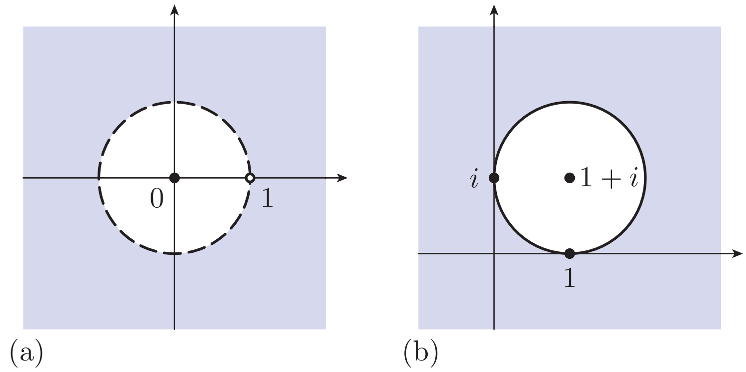


Figure 4.8 (a) $\{z : |z| > 1\}$ (b) $\{z : |z - 1 - i| \geq 1\}$

The set in Figure 4.8(a) comprises those points that lie outside the circle $|z| = 1$, excluding the circle itself (because the boundary is drawn as a broken curve). Points in this set lie at a distance greater than 1 from 0, so the set is made up of points z that satisfy the inequality

$$|z| > 1.$$

The set in Figure 4.8(b) is made up of those points that lie outside the circle centred at $1 + i$ of radius 1, which, as you saw earlier, has equation $|z - 1 - i| = 1$. In this case, the set includes the circle itself, because the boundary is drawn as an unbroken curve. Points in this set lie at a distance greater than or equal to 1 from $1 + i$, so the set is made up of points z that satisfy the inequality

$$|z - 1 - i| \geq 1.$$

In both these figures, the centres of the circles are marked by solid dots. We use solid dots rather than hollow dots, even though the centres do not belong to the sets, because hollow dots are reserved for points on the boundary of a set that are excluded from that set.

Figure 4.9 introduces a new type of set called an *annulus* (the plural is *annuli*). An **annulus** is the set between two concentric circles, possibly including one or both of the boundary circles.

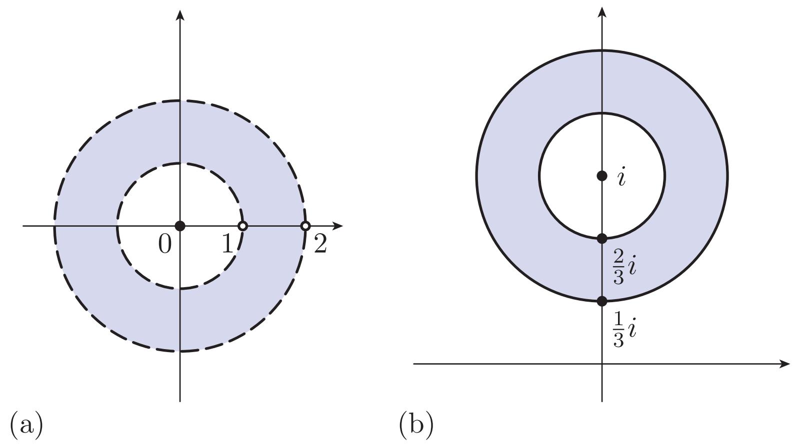


Figure 4.9 (a) Open annulus $\{z : 1 < |z| < 2\}$ (b) Closed annulus $\{z : \frac{1}{3} \leq |z - i| \leq \frac{2}{3}\}$

The annulus in Figure 4.9(a) consists of those points that lie strictly between the circles given by the equations $|z| = 1$ and $|z| = 2$. Both boundary circles are excluded, so the annulus is the set of points z that satisfy the inequalities

$$1 < |z| < 2.$$

An annulus that excludes both its boundary circles is called an *open annulus*.

The annulus in Figure 4.9(b) has boundary circles given by the equations $|z - i| = \frac{1}{3}$ and $|z - i| = \frac{2}{3}$. This time, both circles are included in the set, so this annulus is the set of points z that satisfy the inequalities

$$\tfrac{1}{3} \leq |z - i| \leq \tfrac{2}{3}.$$

An annulus that includes both its boundary circles is called a *closed annulus*.

Definitions

An **open annulus** is a set of the form

$$\{z : r_1 < |z - \alpha| < r_2\},$$

and a **closed annulus** is a set of the form

$$\{z : r_1 \leq |z - \alpha| \leq r_2\},$$

where $\alpha \in \mathbb{C}$ is the centre of the annulus and $r_2 > r_1 > 0$ are the radii of the boundary circles.

Some annuli include one boundary circle but not the other; such annuli are not referred to as either open or closed.

Figure 4.10 shows two **punctured discs**; these are discs from which the centre points have been removed. In each diagram, the centre point is indicated by a hollow dot, because it lies on the boundary of the set, but is not included in the set.

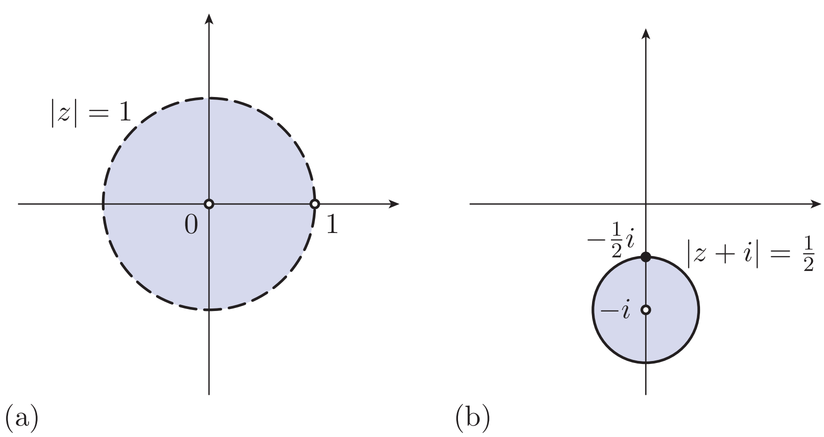


Figure 4.10 Punctured discs: (a) $\{z : 0 < |z| < 1\}$, (b) $\{z : 0 < |z + i| \leq \frac{1}{2}\}$

Figure 4.10(a) is a punctured open disc. It consists of points in the disc $\{z : |z| < 1\}$, except the centre 0. Therefore this punctured disc is the set of points z that satisfy the inequalities

$$0 < |z| < 1.$$

Figure 4.10(b) is a punctured closed disc. It is the set of points z that satisfy the inequalities

$$0 < |z + i| \leq \tfrac{1}{2}.$$

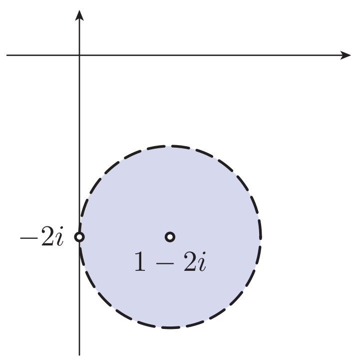
The next exercise gives you practice at sketching sets related to circles and discs.

Exercise 4.4

(a) Sketch the following sets.

(i) $\left\{z : |z + i| > \frac{1}{2}\right\}$
(ii) $\left\{z : \frac{1}{2} \leq |z + 1| < 2\right\}$
(iii) $\{z : 2 \leq |z + 2 - 3i| \leq 3\}$

(b) Use set notation to describe the set shaded in the following figure, which is a punctured disc with centre $1 - 2i$.



Each of the two diagrams in Figure 4.11 displays a *ray* or *half-line*, which is half a straight line with its endpoint missing (indicated by a hollow dot).

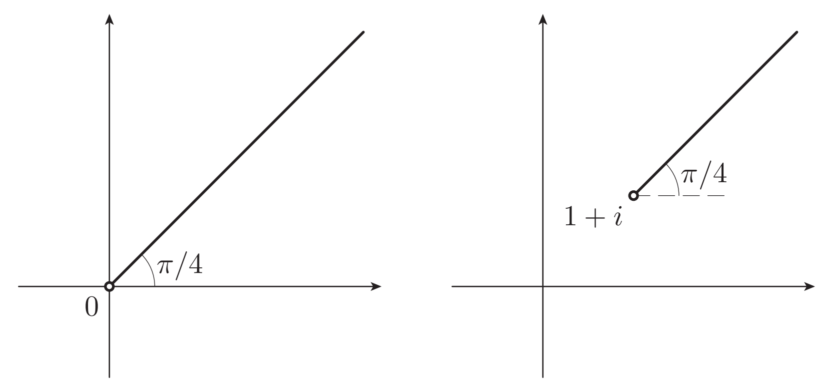


Figure 4.11 Rays: (a) $\{z : \operatorname{Arg} z = \pi/4\}$, (b) $\{z : \operatorname{Arg}(z - 1 - i) = \pi/4\}$

The ray in Figure 4.11(a) comprises those points of the complex plane with principal argument equal to $\pi/4$, so it has equation

$$\operatorname{Arg} z = \pi/4.$$

For Figure 4.11(b), we see that a point z lies on this ray if and only if the point $z - (1 + i)$ lies on the ray in Figure 4.11(a). As $z - (1 + i) = z - 1 - i$, it follows that the equation of Figure 4.11(b) is

$$\operatorname{Arg}(z - 1 - i) = \pi/4.$$

More generally, we have the following definition of a ray.

Definition

A **ray** or **half-line** is a set of the form

$$\{z : \operatorname{Arg}(z - \alpha) = \theta\},$$

where $\alpha \in \mathbb{C}$ and $-\pi < \theta \leq \pi$.

Figure 4.12 shows two examples of sets that we call *sectors*. A **sector** is a set bounded by two rays that share a common endpoint. The sector may or may not include one or both of the boundary rays.

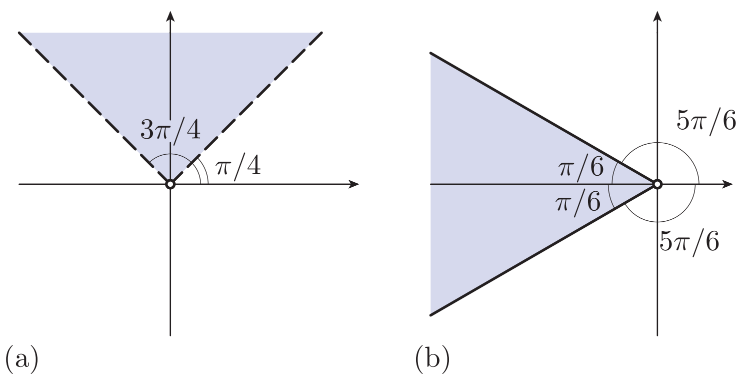


Figure 4.12 Sectors: (a) $\{z : \pi/4 < \operatorname{Arg} z < 3\pi/4\}$, (b) $\{z : |\operatorname{Arg} z| \geq 5\pi/6\}$

The sector in Figure 4.12(a) consists of all points whose principal argument lies strictly between $\pi/4$ and $3\pi/4$, so it is the set of points z for which

$$\pi/4 < \operatorname{Arg} z < 3\pi/4.$$

This set is called an *open sector* because the boundary is excluded.

The sector in Figure 4.12(b) is made up of those points whose principal argument is either less than or equal to $-5\pi/6$, or greater than or equal to $5\pi/6$. That is, the sector contains those points z that satisfy either

$$\operatorname{Arg} z \leq -5\pi/6 \quad \text{or} \quad \operatorname{Arg} z \geq 5\pi/6.$$

We can write these two inequalities as the single inequality

$$|\operatorname{Arg} z| \geq 5\pi/6.$$

The sector in Figure 4.12(b) is *not* called a 'closed sector' because even though the boundary rays are included in the sector, the point 0, which also lies on the boundary, is excluded.

The four quadrants of the complex plane (defined in Subsection 2.1) are open sectors. For example, the upper-right quadrant is the set

$$\{z : 0 < \operatorname{Arg} z < \pi/2\}.$$

Two more sectors are shown in Figure 4.13. In each of these sectors the boundary rays meet at vertices away from the origin.

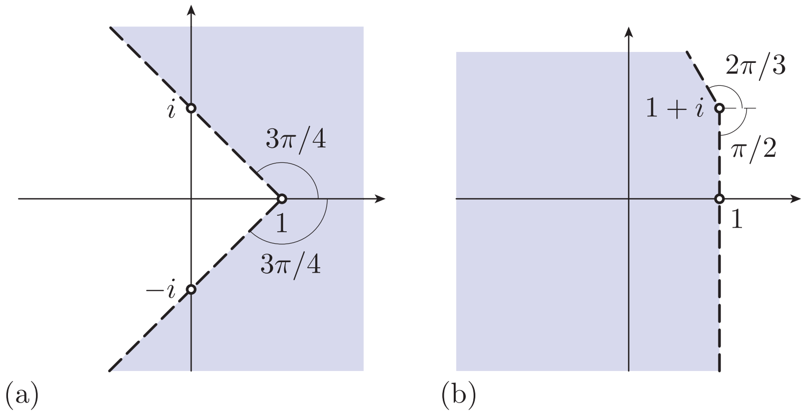


Figure 4.13 Open sectors: (a) $\{z : |\operatorname{Arg}(z-1)| < 3\pi/4\}$, (b) $\{z : \operatorname{Arg}(z-1-i) < -\pi/2 \text{ or } \operatorname{Arg}(z-1-i) > 2\pi/3\}$

To find the inequality for the sector in Figure 4.13(a), first consider the set that is obtained by translating this sector by one unit to the left. This new set is itself a sector, comprising points z that satisfy the inequality

$$|\operatorname{Arg} z| < 3\pi/4.$$

Under the translation, the point z is moved to the point $z-1$, so it follows that the original sector comprises points z that satisfy the inequality

$$|\operatorname{Arg}(z-1)| < 3\pi/4.$$

Reasoning in a similar way, we find that the sector in Figure 4.13(b) consists of points z that satisfy either

$$\operatorname{Arg}(z-1-i) < -\pi/2 \quad \text{or} \quad \operatorname{Arg}(z-1-i) > 2\pi/3.$$

Both sectors in Figure 4.13 are open sectors. The general definition of an open sector is as follows.

Definition

An **open sector** is a set of one of the forms

$$\{z : a < \operatorname{Arg}(z-\alpha) < b\}$$

or

$$\{z : \operatorname{Arg}(z-\alpha) < a \text{ or } \operatorname{Arg}(z-\alpha) > b\},$$

where $\alpha \in \mathbb{C}$ and $-\pi < a < b \leq \pi$.

The complex number α in this definition is the location of the vertex of the sector, and a and b are real numbers that determine the angles of the boundary rays.

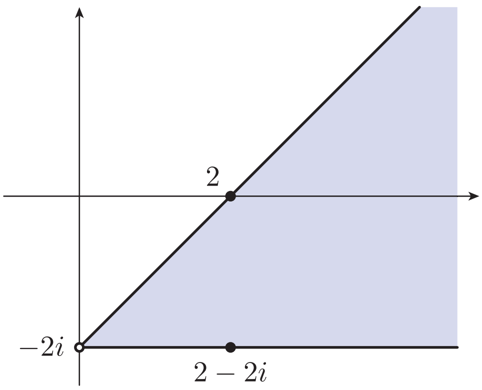
Exercise 4.5

(a) Sketch the following sets.

(i) $\{z : \operatorname{Arg} z = -2\pi/3\}$ (ii) $\{z : \operatorname{Arg}(z - i) = 3\pi/4\}$

(iii) $\{z : |\operatorname{Arg} z| < 2\pi/3\}$

(b) Use set notation to describe the set shaded in the following figure.



We can use standard operations on sets to create new subsets of the complex plane from old ones. Three of these operations are defined below.

Definitions

Let A and B be subsets of the complex plane.

The **union** of A and B is

$$A \cup B = \{z : z \in A \text{ or } z \in B\}.$$

The **intersection** of A and B is

$$A \cap B = \{z : z \in A \text{ and } z \in B\}.$$

The **difference** of A and B is

$$A - B = \{z : z \in A \text{ and } z \notin B\}.$$

Remarks

1. The set $A \cap B$ is also commonly written as $\{z : z \in A,\ z \in B\}$; the comma takes the place of the word 'and'.
2. The difference $A - B$ should be read as 'A minus B'.

These operations are illustrated by the Venn diagrams in Figure 4.14; these are abstract depictions of the operations. (They are *not* sketches in the complex plane.) The sets A and B are enclosed by the circles in each diagram, and the shaded parts represent $A \cup B$, $A \cap B$ and $A - B$, respectively.

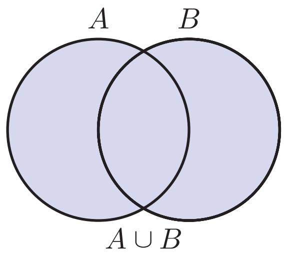


Figure 4.14 Venn diagrams

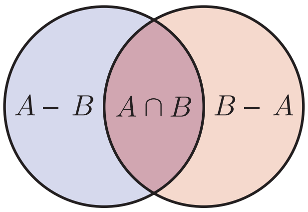


Figure 4.15 Disjoint sets $A - B$, $A \cap B$ and $B - A$

Notice that, as shown in Figure 4.15, the three sets $A - B$, $A \cap B$ and $B - A$ are mutually disjoint, meaning that no two of them have any points in common, and their union is $A \cup B$.

We now examine the effects of these operations on the subsets A and B of the complex plane shown in Figure 4.16.

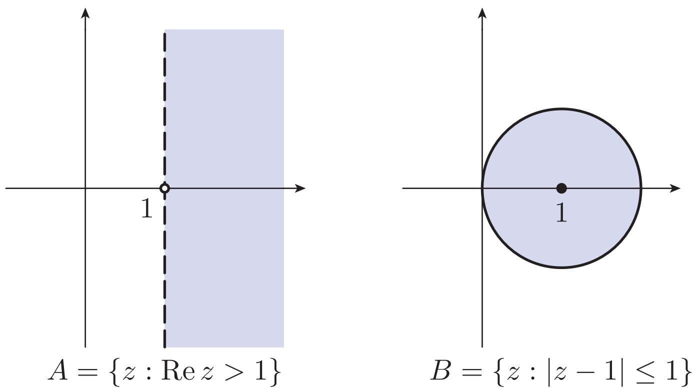


Figure 4.16 An open half-plane A and a closed disc B

The sets $A \cup B$, $A \cap B$ and $A - B$ are displayed in Figure 4.17.

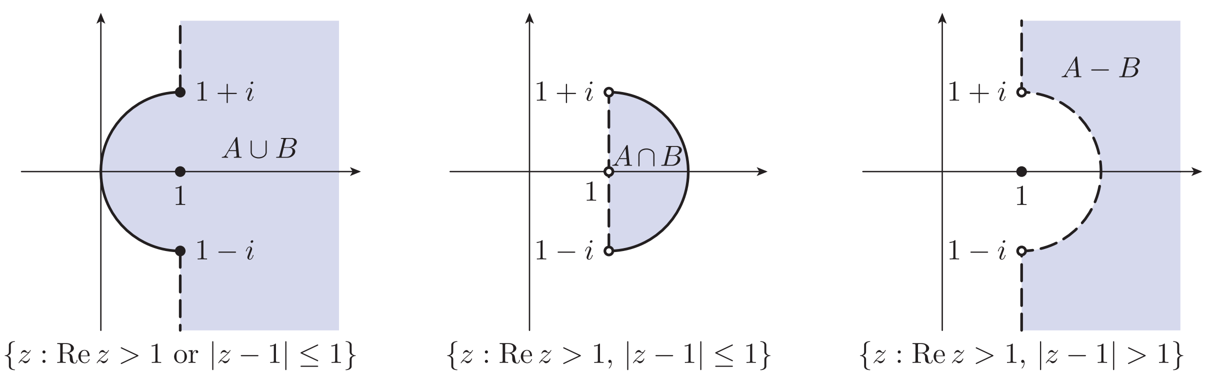


Figure 4.17 The sets $A \cup B$, $A \cap B$ and $A - B$

Since A is determined by the single condition $\operatorname{Re} z > 1$, and B is determined by the single condition $|z-1| \leq 1$, the set $A \cup B$ consists of points z for which $\operatorname{Re} z > 1$ or $|z-1| \leq 1$. The set $A \cap B$ is made up of points that satisfy both conditions, $\operatorname{Re} z > 1$ and $|z-1| \leq 1$. Finally, the set $A - B$ comprises points z for which the condition $\operatorname{Re} z > 1$ is true and the condition $|z-1| \leq 1$ is false. That is, $A - B$ is determined by the inequalities $\operatorname{Re} z > 1$ and $|z-1| > 1$.

We finish here by discussing *complements* of subsets of the complex plane.

Definition

The **complement** of a subset A of the complex plane is the set $\mathbb{C} - A$ of all the points of $\mathbb{C}$ that are *not* in A.

We have already seen some examples of complements. For instance, Figure 4.8(a) illustrates the complement of a closed disc, and Figure 4.8(b) illustrates the complement of an open disc. Figure 4.18 provides two more examples.

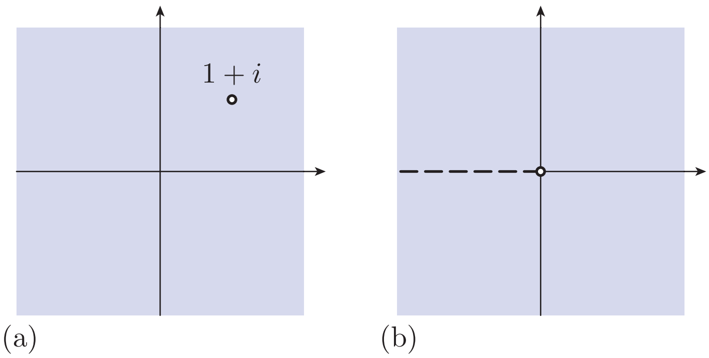


Figure 4.18 (a) A punctured plane $\mathbb{C} - \{1+i\} = \{z : |z-1-i| > 0\}$
(b) The cut plane $\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\} = \{z : |\operatorname{Arg} z| < \pi\}$

Figure 4.18(a) is the complex plane with a single point removed; such sets are called **punctured planes**. The punctured plane in Figure 4.18(a) is the complement of the single-point set $\{1+i\}$. Figure 4.18(b) is the complex plane with the negative real axis and 0 removed. This set can be described by the equation $|\operatorname{Arg} z| < \pi$; we call it a *cut plane*.

Definition

A **cut plane** is the complex plane $\mathbb{C}$ with a half-line from the origin and the origin itself removed.

In particular, the set $\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$ is a cut plane, and this set can also be specified as $\{z : |\operatorname{Arg} z| < \pi\}$.

Note that some texts use the phrase *slit plane* instead of *cut plane*.

The next exercise gives you practice at applying set operations.

Exercise 4.6

Let

$$A = \{z : |\operatorname{Re} z| < 1\} \quad \text{and} \quad B = \{z : |z| \leq 2\}.$$

(a) Sketch the following sets (on separate diagrams).

(i) A (ii) B (iii) $A \cup B$ (iv) $A \cap B$ (v) $A - B$
(vi) $\mathbb{C} - A$

(b) Complete the statement $\mathbb{C} - (A \cup B) = \{z : \qquad\qquad \}$.

We summarise the various conventions for sketching subsets of $\mathbb{C}$.

Sketching conventions

- The interior of a set is shown by shading.
- Boundary curves that belong to the set are drawn unbroken.
- Boundary curves that do not belong to the set are drawn broken.
- Distinguished boundary points that belong to the set are drawn as solid dots (small, filled-in circles).
- Distinguished boundary points that do not belong to the set are drawn as hollow dots (small, empty circles).

When sketching, there is some latitude as to which points you should mark on your sketch and whether you should calculate the complex numbers corresponding to those points. As a general rule, you should always mark distinguished boundary points of your sketch, such as points where two boundary curves meet at a corner, or distinguished points that are excluded from the set. You should also include points to help specify exactly what the sketch represents; for example, you may include some intersection points of boundary curves with the axes. However, do not include too many points, or your diagram will appear cluttered. Usually, you should write down the complex number corresponding to a point that you have included, provided that the complex number is relatively straightforward to determine.

These conventions will remain in force throughout the module, although we will not always include shading. When sketching sets by hand you could consider replacing shading by hatching, as shown in Figure 4.19.

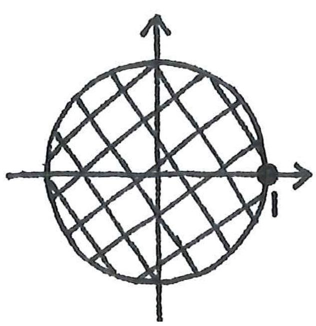


Figure 4.19 A disc filled with hatching

Exercise 4.7

Sketch the following sets, using the sketching conventions.

(a) $\{z : \operatorname{Im} z > 0\}$

(b) $\{z : |z+1| \leq 1\}$

(c) $\{z : 0 < |z+1+2i| < 1\}$

(d) $\{z : |\operatorname{Arg}(z+1-i)| < \pi/3\}$

(e) $\{z : |z-1| \leq |z-2|\}$

(f) $\mathbb{C} - \{z : \operatorname{Re} z \geq 1\}$

(g) $\{z : \operatorname{Im} z > 0\} - \{z : |z+1| \leq 1\}$

(h) $\{z : \operatorname{Arg} z = \pi/6\} \cup \{z : \operatorname{Arg}(z - \sqrt{3} - i) = 0\}$

(i) $\{z : \operatorname{Arg} z = \pi/6\} \cap \{z : \operatorname{Arg}(z - \sqrt{3} - i) = 0\}$

(*Hint*: For part (e), interpret the inequality in terms of distances.)

Further exercises

Exercise 4.8

Sketch the following sets, using the sketching conventions.

(a) $\{z : |\operatorname{Re} z| < 1, |\operatorname{Im} z| < 1\}$

(b) $\{z : |z-i| \leq 2, |z| \geq 1\}$

(c) $\{z : \operatorname{Re} z + 2\operatorname{Im} z + 3 > 0\}$

(d) $\{z : \operatorname{Re} z \geq 0\} \cup \{z : \operatorname{Im} z > 0\}$

(e) $\{z : |z| > 1, |\operatorname{Arg} z| \leq \pi/4\}$

(f) $\{z : |z+1+2i| \leq 1\}$

(g) $\{z : \operatorname{Re} z > 1, |z-i| < 2\}$

(h) $\{z : |z+i| < |z+2i|\}$

(i) $\{z : |z| < 3\} - \{z : |z| \leq 2\}$

(j) $\mathbb{C} - \{z : z^2 + z - 2 = 0\}$

5 Proving inequalities

After working through this section, you should be able to:

- use the rules for rearranging inequalities and the rules for obtaining new inequalities from old ones
- prove inequalities involving the moduli of complex numbers by using various forms of the Triangle Inequality.

5.1 Rules for rearranging inequalities

In Section 4 we used equalities and inequalities to define subsets of the complex plane. In this section you will see how to prove new inequalities by deducing them from simpler known inequalities (such as $|z| \geq 0$, which holds for all z) using various rules. We begin by reminding you of the rules for rearranging a given inequality into an *equivalent* form; such equivalent inequalities are linked by the symbol '$\iff$', which may be read as 'is equivalent to' or 'if and only if'.

Rules for Rearranging Inequalities

For all $a, b, c \in \mathbb{R}$, the following rules apply.

Rule 1 $a < b \iff b - a > 0$.

Rule 2 $a < b \iff a + c < b + c$.

Rule 3 If $c > 0$, then $a < b \iff ac < bc$.
If $c < 0$, then $a < b \iff ac > bc$.

Rule 4 If $a, b > 0$, then $a < b \iff \dfrac{1}{a} > \dfrac{1}{b}$.

Rule 5 If $a, b \geq 0$ and $p > 0$, then $a < b \iff a^p < b^p$.

Rule 6 $|a| < b \iff -b < a < b$.

There are corresponding versions of Rules 1–6 in which the strict inequality '$<$' is replaced by the weak inequality '$\leq$'.

The next two rules can be used to deduce new inequalities from given ones. Here, however, the new inequalities are not equivalent to the old ones, since the old inequalities cannot be deduced from the new ones. Such deductions are written using the symbol '$\implies$', which may be read as 'implies'.

Transitive Rule

For all $a, b, c \in \mathbb{R}$,

$$a < b \text{ and } b < c \implies a < c.$$

For example, if $x < 2$, then $x < 3$ (because $2 < 3$).

Combination Rules for Inequalities

For all $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$, the following rules apply.

(a) **Sum Rule** $a + c < b + d$.

(b) **Product Rule** $ac < bd$ (provided that $a, c \geq 0$).

For example, if $0 \leq n < 5$, then (since $2 < 3$),

$$n + 2 < 5 + 3 = 8 \quad \text{and} \quad 2n < 3 \times 5 = 15.$$

There are also weak and weak/strict versions of the Transitive Rule and the Combination Rules, which you should be able to work out as they arise.

The following example illustrates how the various rules are used in practice.

Example 5.1

Prove that

$$2r^2 > (r+1)^2, \quad \text{for } r \geq 3.$$

Solution

We rearrange the given inequality in order to find an equivalent, but simpler one:

$$\begin{aligned} 2r^2 > (r+1)^2 &\iff 2 > \left(\frac{r+1}{r}\right)^2 && \text{(Rule 3)} \\ &\iff \sqrt{2} > 1 + \frac{1}{r} && \text{(Rule 5)} \\ &\iff \sqrt{2} - 1 > \frac{1}{r} && \text{(Rule 2)} \\ &\iff r > \frac{1}{\sqrt{2}-1} = \sqrt{2} + 1 && \text{(Rule 4).} \end{aligned}$$

Since we are given that $r \geq 3$, and $3 > \sqrt{2} + 1 = 2.414\ldots$, it follows from the Transitive Rule that $r > \sqrt{2} + 1$. Therefore the final inequality is true, so the first inequality must be true for $r \geq 3$ also.

Remarks

1. Example 5.1 could be solved, alternatively, by using Rule 1 to obtain the equivalent inequality $r^2 - 2r - 1 > 0$ and then completing the square. There is often more than one way to deal with a given inequality.
2. In future we will not usually indicate which rule for rearranging a given inequality is being used.

Exercise 5.1

Prove that

$$\frac{3r}{r^2+2} < 1, \quad \text{for } r > 2.$$

5.2 The Triangle Inequality

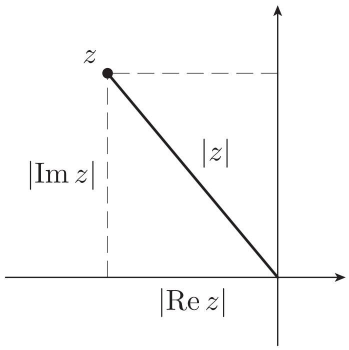


Figure 5.1 Right-angled triangle with hypotenuse $|z|$

Many inequalities have a geometric interpretation. For example, the two inequalities

$$|x| \leq \sqrt{x^2+y^2} \quad \text{and} \quad |y| \leq \sqrt{x^2+y^2}$$

can be used to represent the statement that, in a right-angled triangle, the hypotenuse is the longest side. They can be written in complex form as follows (see Figure 5.1).

$$|\mathrm{Re}\, z| \leq |z| \quad \text{and} \quad |\mathrm{Im}\, z| \leq |z|.$$

Or, equivalently, they can be written as

$$-|z| \leq \mathrm{Re}\, z \leq |z| \quad \text{and} \quad -|z| \leq \mathrm{Im}\, z \leq |z|.$$

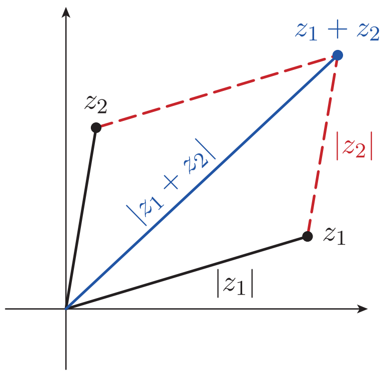


Figure 5.2 Demonstration of the Triangle Inequality

Another elementary fact from plane geometry is that the length of any side of a triangle is less than or equal to the sum of the lengths of the other two sides. If z_1, z_2 are complex numbers, then 0, z_1 and $z_1 + z_2$ form the vertices of a triangle (see Figure 5.2) with side lengths $|z_1|$, $|z_2|$ and $|z_1 + z_2|$, so

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

This is one form of an inequality called the Triangle Inequality, which will be used frequently throughout the module.

Theorem 5.1 Triangle Inequality

If $z_1, z_2 \in \mathbb{C}$, then

(a) $|z_1 + z_2| \leq |z_1| + |z_2|$ (usual form)

(b) $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$ (backwards form).

An immediate consequence of the second part of this theorem is that

$$|z_1 - z_2| \geq |z_1| - |z_2| \quad \text{and} \quad |z_1 - z_2| \geq |z_2| - |z_1|.$$

Proof Although part (a) follows from plane geometry, we give a proof using complex numbers that illustrates the use of several results about complex conjugates from this unit.

We have

$$\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2)\overline{(z_1 + z_2)} && \text{(Theorem 2.1(c))}\\
&= (z_1 + z_2)(\overline{z_1} + \overline{z_2}) && \text{(Theorem 1.1(b)(i))}\\
&= z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_2}\\
&= |z_1|^2 + z_1\overline{z_2} + \overline{z_1\overline{z_2}} + |z_2|^2 && \text{(Theorems 2.1(c), 1.1(a)(iii)}\\
& && \text{and 1.1(b)(iii))}
\end{aligned}$$

$$
\begin{aligned}
&= |z_1|^2 + 2\,\mathrm{Re}(z_1\overline{z_2}) + |z_2|^2 && \text{(Theorem 1.1(a)(i))}\\
&\le |z_1|^2 + 2|z_1\overline{z_2}| + |z_2|^2 && (\text{since } |\mathrm{Re}\, z| \le |z|)\\
&= |z_1|^2 + 2|z_1||z_2| + |z_2|^2 && \text{(Theorem 2.1(b),(e))}\\
&= (|z_1| + |z_2|)^2
\end{aligned}
$$

and so part (a) follows.

Part (b) can be proved by a similar method; alternatively, note that

$$|z_1| = |z_1 - z_2 + z_2| \le |z_1 - z_2| + |z_2|,$$

by part (a), so

$$|z_1 - z_2| \ge |z_1| - |z_2|. \tag{5.1}$$

Similarly,

$$|z_2 - z_1| \ge |z_2| - |z_1|,$$

and since $|z_2 - z_1| = |z_1 - z_2|$, we see that

$$|z_1 - z_2| \ge |z_2| - |z_1|. \tag{5.2}$$

Part (b) then follows from inequalities (5.1) and (5.2). ■

The backwards form of the Triangle Inequality also has a useful geometric interpretation, concerning the two circles centred at 0 through z_1 and z_2. It says that the distance from z_2 to z_1 is at least as large as the difference between the radii of these circles, as shown in Figure 5.3 for the case $|z_1| > |z_2|$.

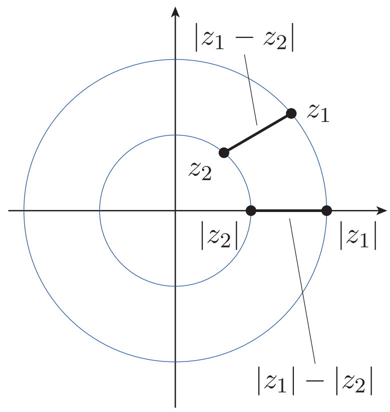


Figure 5.3 Demonstration that $|z_1 - z_2| \ge |z_1| - |z_2|$

Several other versions of the Triangle Inequality are given in the following corollary. Each is a variant of one of the forms of the Triangle Inequality.

Corollary

If $z, z_1, z_2, \ldots, z_n \in \mathbb{C}$, then

(a) $|z| \le |\mathrm{Re}\, z| + |\mathrm{Im}\, z|$

(b) $|z_1 - z_2| \le |z_1| + |z_2|$

(c) $|z_1 + z_2| \ge \big||z_1| - |z_2|\big|$

(d) $|z_1 \pm z_2 \pm \cdots \pm z_n| \le |z_1| + |z_2| + \cdots + |z_n|$

(e) $|z_1 \pm z_2 \pm \cdots \pm z_n| \ge |z_1| - |z_2| - \cdots - |z_n|$.

Parts (a), (b) and (d) are variants of the usual form of the Triangle Inequality, whereas parts (c) and (e) are variants of the backwards form.

Proof Part (a) is obtained by taking $z_1 = \mathrm{Re}\, z$ and $z_2 = i\,\mathrm{Im}\, z$ in the usual form of the Triangle Inequality.

Parts (b) and (c) are obtained by substituting $-z_2$ for z_2 in Theorem 5.1.

Parts (d) and (e) are obtained from parts (b) and (c) of this corollary and Theorem 5.1 by applying the Principle of Mathematical Induction – we omit the details. ■

The Triangle Inequality can be used to obtain estimates (also known as bounds) for the modulus of a complex expression involving z when we know that z lies in a certain set (such as a circle). The next example includes some typical applications. As indicated in the solutions to the example, it is not usual to refer explicitly to any of the variants (in the corollary) of the Triangle Inequality. However, use of the backwards form should be distinguished.

Example 5.2

(a) Prove the following inequalities.

(i) $|z^2 - 4z - 3| \leq 15, \quad \text{for } |z| = 2$

(ii) $|z^2 - 7| \geq 3, \quad \text{for } |z| = 2$

(iii) $|z^2 + 2| \geq 2, \quad \text{for } |z| = 2$

(b) Find a number M such that

$$\left|\frac{z^2 - 4z - 3}{(z^2 - 7)(z^2 + 2)}\right| \leq M, \quad \text{for } |z| = 2.$$

Solution

(a) (i) By the Triangle Inequality,

$$\begin{aligned}|z^2 - 4z - 3| &\leq |z^2| + |-4z| + |-3| \\ &= |z|^2 + 4|z| + 3;\end{aligned}$$

so, for $|z| = 2$,

$$|z^2 - 4z - 3| \leq 4 + 8 + 3 = 15.$$

(ii) By the backwards form of the Triangle Inequality,

$$|z^2 - 7| \geq \big||z|^2 - 7\big|;$$

so, for $|z| = 2$,

$$|z^2 - 7| \geq |4 - 7| = 3.$$

(iii) By the backwards form of the Triangle Inequality,

$$|z^2 + 2| \geq \big||z|^2 - 2\big|;$$

so, for $|z| = 2$,

$$|z^2 + 2| \geq |4 - 2| = 2.$$

(b) From part (a) we have, for $|z| = 2$,

$$|z^2 - 4z - 3| \leq 15, \quad |z^2 - 7| \geq 3, \quad |z^2 + 2| \geq 2.$$

Now

$$\begin{aligned}\left|\frac{z^2 - 4z - 3}{(z^2 - 7)(z^2 + 2)}\right| &= \frac{|z^2 - 4z - 3|}{|z^2 - 7| \times |z^2 + 2|} \\ &= |z^2 - 4z - 3| \times \frac{1}{|z^2 - 7|} \times \frac{1}{|z^2 + 2|}.\end{aligned}$$

So, for $|z| = 2$, using the inequalities from part (a), we have

$$\left|\frac{z^2 - 4z - 3}{(z^2 - 7)(z^2 + 2)}\right| \leq 15 \times \frac{1}{3} \times \frac{1}{2} = \frac{5}{2},$$

because

$$|z^2 - 7| \geq 3 \implies 1/|z^2 - 7| \leq 1/3$$

and

$$|z^2 + 2| \geq 2 \implies 1/|z^2 + 2| \leq 1/2.$$

Thus we can take $M = 5/2$.

Remarks

1. Example 5.2(b) illustrates the fact that to obtain an upper estimate for a quotient, we need an upper estimate (in this case, 15) for the numerator and a *lower estimate* (in this case, 3×2) for the denominator.
2. Equality is attained in the inequality

   $$|z^2 + 2| \geq 2, \quad \text{for } |z| = 2,$$

   when $z = 2i$ (or $z = -2i$), because

   $$|(2i)^2 + 2| = |-4 + 2| = 2.$$

   In contrast, it is not possible to attain equality in the inequality

   $$|z^2 - 4z - 3| \leq 15, \quad \text{for } |z| = 2,$$

   because it can be shown that the inequality remains true if 15 is replaced by certain smaller numbers, the smallest possible one of which is $7\sqrt{7/3} = 10.69\ldots$.

Exercise 5.2

Prove the following inequalities.

(a) $\displaystyle \frac{1}{7} \leq \left| \frac{1}{3 + 4z^2} \right| \leq 1, \quad \text{for } |z| = 1$

(b) $\displaystyle 2 \leq \left| \frac{z^3 + 2z + 1}{z^2 + 1} \right| \leq \frac{17}{4}, \quad \text{for } |z| = 3$

Further exercises

Exercise 5.3

For $|z| = 2$, find an upper estimate for each of the following moduli.

(a) $|z + 3|$ (b) $|z - 4i|$ (c) $|3z + 2|$ (d) $|3z^2 - 5|$

(e) $|z^2 + z + 1|$

Exercise 5.4

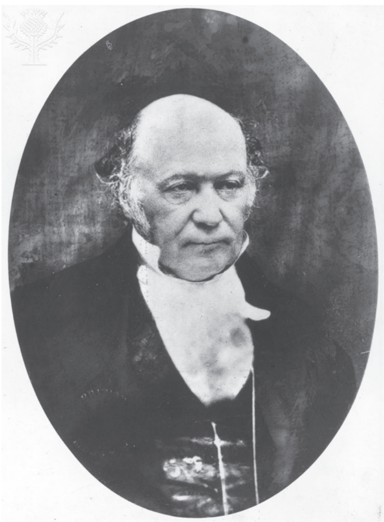
For $|z| = 5$, find a positive lower estimate for each of the following moduli.

(a) $|z - 2|$ (b) $|z + 3i|$ (c) $|z - 7|$ (d) $|2z - 7|$

Exercise 5.5

Find positive numbers m and M such that

$$m \leq \left|\frac{z^3 + 1}{z^3 - 1}\right| \leq M, \quad \text{for } |z| = 4.$$



William Rowan Hamilton

Quaternions

As you have seen, the complex numbers are formed by adjoining to the real numbers a new symbol i with the property $i^2 = -1$. In 1843 the Irish mathematician William Rowan Hamilton (1805–1865), whom you met in the Introduction, realised that by adjoining to the real numbers *three* new symbols i, j and k with the properties

$$i^2 = j^2 = k^2 = ijk = -1$$

we obtain a collection of numbers $a + bi + cj + dk$ (where $a, b, c, d \in \mathbb{R}$) with almost all of the algebraic properties of a field. Hamilton called these numbers **quaternions**. The quaternions are peculiar in that multiplication of quaternions is not commutative; for example, it can be proved that

$$ij = k, \quad \text{whereas} \quad ji = -k.$$

Since they satisfy all the other properties of a field (all but property M5 from the 'Arithmetic in $\mathbb{C}$' table after Theorem 1.1) the quaternions can be described as a **non-commutative field**.

You have learned in this unit how the algebra of complex arithmetic is complemented by two-dimensional geometry in the complex plane. Quaternions are four-dimensional numbers, and they can be used to represent geometric objects in three and four dimensions. They play an important role in computer graphics, where quaternion algebra is used to manipulate three-dimensional images. They also feature in quantum mechanics, as an algebraic tool for representing the spin of elementary particles.

Solutions to exercises

Solution to Exercise 1.1

(a) **(i)** $(2+i)+3i(-1+3i) = 2+i-3i-9$
$$= -7-2i$$

(ii) $(2+i)(-1+3i) = -2+6i-i-3$
$$= -5+5i$$

(iii) $(-1+3i)(-1-3i) = 1+3i-3i+9 = 10$

(b) By part (a)(i), $\operatorname{Re} z = -7$ and $\operatorname{Im} z = -2$.

Solution to Exercise 1.2

(a) $(x_1+iy_1)+(x_2+iy_2)$
$$= (x_1+x_2)+i(y_1+y_2)$$

(b) $(x_1+iy_1)-(x_2+iy_2)$
$$= (x_1-x_2)+i(y_1-y_2)$$

(c) $(x_1+iy_1)(x_2+iy_2)$
$$= x_1x_2+ix_1y_2+iy_1x_2-y_1y_2$$
$$= (x_1x_2-y_1y_2)+i(x_1y_2+y_1x_2)$$

(d) $(x+iy)(x-iy) = x^2-ixy+iyx+y^2$
$$= x^2+y^2$$

Solution to Exercise 1.3

(a) **(i)** $\dfrac{1}{i} = \dfrac{-i}{i\times(-i)} = \dfrac{-i}{1} = -i$

(ii) $\dfrac{1}{1+i} = \dfrac{1-i}{(1+i)(1-i)} = \dfrac{1-i}{1+1} = \dfrac{1}{2}-\dfrac{1}{2}i$

(iii) $\dfrac{1+2i}{2+3i} = \dfrac{(1+2i)(2-3i)}{(2+3i)(2-3i)}$
$$= \frac{8+i}{4+9} = \frac{8}{13}+\frac{1}{13}i$$

(b)
$$\frac{x_1+iy_1}{x_2+iy_2} = \frac{(x_1+iy_1)(x_2-iy_2)}{(x_2+iy_2)(x_2-iy_2)}$$
$$= \frac{(x_1x_2+y_1y_2)+i(y_1x_2-x_1y_2)}{x_2^2+y_2^2}$$
$$= \left(\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}\right)+i\left(\frac{y_1x_2-x_1y_2}{x_2^2+y_2^2}\right)$$

Solution to Exercise 1.4

Theorem 1.1(b)(i)

Let $z_1 = x_1+iy_1$ and $z_2 = x_2+iy_2$. Then
$$z_1+z_2 = (x_1+x_2)+i(y_1+y_2),$$
so
$$\overline{z_1+z_2} = (x_1+x_2)-i(y_1+y_2).$$
Also, $\overline{z_1} = x_1-iy_1$ and $\overline{z_2} = x_2-iy_2$, so
$$\overline{z_1}+\overline{z_2} = (x_1+x_2)-i(y_1+y_2)$$
$$= \overline{z_1+z_2},$$
as required.

Theorem 1.1(b)(iv)

Let $z_1 = x_1+iy_1$ and $z_2 = x_2+iy_2$. Then
$$\frac{z_1}{z_2} = \left(\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}\right)+i\left(\frac{y_1x_2-x_1y_2}{x_2^2+y_2^2}\right),$$
by Exercise 1.3(b), so
$$\overline{(z_1/z_2)} = \left(\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}\right)-i\left(\frac{y_1x_2-x_1y_2}{x_2^2+y_2^2}\right).$$
Also, $\overline{z_1} = x_1-iy_1$ and $\overline{z_2} = x_2-iy_2$, so
$$\frac{\overline{z_1}}{\overline{z_2}} = \left(\frac{x_1x_2+(-y_1)(-y_2)}{x_2^2+(-y_2)^2}\right)$$
$$+i\left(\frac{(-y_1)x_2-x_1(-y_2)}{x_2^2+(-y_2)^2}\right)$$
$$= \left(\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}\right)-i\left(\frac{y_1x_2-x_1y_2}{x_2^2+y_2^2}\right)$$
$$= \overline{(z_1/z_2)},$$
as required.

Solution to Exercise 1.5

(a) $(z_1+z_2)^3$
$$= (z_1+z_2)(z_1^2+2z_1z_2+z_2^2)$$
$$= z_1^3+2z_1^2z_2+z_1z_2^2+z_1^2z_2+2z_1z_2^2+z_2^3$$
$$= z_1^3+3z_1^2z_2+3z_1z_2^2+z_2^3$$

(b) $(z_1-z_2)(z_1^2+z_1z_2+z_2^2)$
$$= z_1^3+z_1^2z_2+z_1z_2^2-z_1^2z_2-z_1z_2^2-z_2^3$$
$$= z_1^3-z_2^3$$

(c) $(z_1+z_2)(z_1^2-z_1z_2+z_2^2)$
$$= z_1^3-z_1^2z_2+z_1z_2^2+z_1^2z_2-z_1z_2^2+z_2^3$$
$$= z_1^3+z_2^3$$

Alternatively, apply part (b) with z_2 replaced by $-z_2$.

Solution to Exercise 1.6

(a) By the Binomial Theorem,

$$\begin{aligned}(1+i)^4 &= 1+4i+6i^2+4i^3+i^4\\ &= 1+4i-6-4i+1 = -4.\end{aligned}$$

(b) By the Binomial Theorem,

$$\begin{aligned}(3+2i)^3 &= 3^3+3\times 3^2\times 2i+3\times 3\times(2i)^2\\ &\qquad +(2i)^3\\ &= 27+54i-36-8i = -9+46i.\end{aligned}$$

Solution to Exercise 1.7

(a) By the Geometric Series Identity,

$$\begin{aligned}&1+(1+i)+(1+i)^2+(1+i)^3\\ &= \frac{1-(1+i)^4}{1-(1+i)}\\ &= \frac{1-(-4)}{-i} \quad \text{(by Exercise 1.6(a))}\\ &= \frac{5\times i}{-i\times i} = 5i.\end{aligned}$$

(b) By the Geometric Series Identity and the hint,

$$\begin{aligned}z^5-i &= z^5-i^5\\ &= (z-i)(z^4+z^3i+z^2i^2+zi^3+i^4)\\ &= (z-i)(z^4+z^3i-z^2-zi+1).\end{aligned}$$

So $z-i$ is one factor.

Solution to Exercise 1.8

z	$\operatorname{Re} z$	$\operatorname{Im} z$	$-z$	$\overline{z}$
$2+3i$	2	3	$-2-3i$	$2-3i$
$-3-i$	-3	-1	$3+i$	$-3+i$
$4i$	0	4	$-4i$	$-4i$
5	5	0	-5	5
0	0	0	0	0

Solution to Exercise 1.9

(a) $i^3 = (i^2)i = -i$

(b) $i^4 = (i^2)(i^2) = 1$

(c) $(1+i)^2 = 1+2i+i^2 = 2i$

(d) $(1-i)^2 = 1-2i+(-i)^2 = -2i$

(e) $\dfrac{1}{1-i} = \dfrac{1+i}{(1-i)(1+i)} = \dfrac{1+i}{2} = \tfrac{1}{2}+\tfrac{1}{2}i$

(f) $\dfrac{1+i}{1-i} = \dfrac{(1+i)(1+i)}{(1-i)(1+i)} = \dfrac{2i}{2} = i$

(g)
$$\begin{aligned}(1+i)^3 &= (1+i)^2(1+i)\\ &= 2i(1+i) = -2+2i\end{aligned}$$

Alternatively, use the Binomial Theorem.

(h) Use the identity $a^2-b^2 = (a+b)(a-b)$:

$$\begin{aligned}&(2+i)^2-(2-i)^2\\ &= ((2+i)+(2-i))((2+i)-(2-i))\\ &= 4\times 2i = 8i.\end{aligned}$$

Alternatively, expand $(2+i)^2$ and $(2-i)^2$ separately.

(i)
$$\begin{aligned}\frac{3+5i}{2-3i} &= \frac{(3+5i)(2+3i)}{(2-3i)(2+3i)}\\ &= \frac{6+19i-15}{4+9} = -\frac{9}{13}+\frac{19}{13}i\end{aligned}$$

(j)
$$\begin{aligned}\frac{3+2i}{1+4i} &= \frac{(3+2i)(1-4i)}{(1+4i)(1-4i)}\\ &= \frac{3-10i+8}{1+16} = \frac{11}{17}-\frac{10}{17}i\end{aligned}$$

(k) By the Binomial Theorem,

$$\begin{aligned}&(3+4i)^4-(3-4i)^4\\ &= (3^4+4\times 3^3\times(4i)+6\times 3^2\times(4i)^2\\ &\qquad +4\times 3\times(4i)^3+(4i)^4)\\ &\quad -(3^4+4\times 3^3\times(-4i)+6\times 3^2\times(-4i)^2\\ &\qquad +4\times 3\times(-4i)^3+(-4i)^4)\\ &= 2(432i-768i) = -672i.\end{aligned}$$

Alternatively, write

$$\begin{aligned}&(3+4i)^4-(3-4i)^4\\ &= ((3+4i)^2-(3-4i)^2)((3+4i)^2+(3-4i)^2),\end{aligned}$$

and then observe that $(3+4i)^2 = -7+24i$ and $(3-4i)^2 = -7-24i$, and simplify.

(l) By the Geometric Series Identity,

$$\begin{aligned}&1+i+i^2+\cdots+i^{10}\\ &= \frac{1-i^{11}}{1-i}\\ &= \frac{1+i}{1-i} = \tfrac{1}{2}(1+i)^2 = i.\end{aligned}$$

(m) By the Geometric Series Identity,

$$\begin{aligned}1 - i + i^2 - \cdots + i^{10} &= \frac{1-(-i)^{11}}{1-(-i)}\\ &= \frac{1-i}{1+i} = \frac{(1-i)(1-i)}{(1+i)(1-i)} = \frac{-2i}{2} = -i.\end{aligned}$$

Alternatively, observe that

$$\overline{i^n} = (\overline{i})^n = (-i)^n = (-1)^n i^n,$$

so by taking the complex conjugate of both sides of the equation

$$1 + i + i^2 + \cdots + i^{10} = i$$

found in part (l), we obtain

$$1 - i + i^2 - \cdots + i^{10} = -i.$$

Solution to Exercise 1.10

z	$\operatorname{Re} z$	$\operatorname{Im} z$	$\overline{z}$
(a) $-i$	0	-1	i
(e) $\frac{1}{2} + \frac{1}{2}i$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2} - \frac{1}{2}i$
(g) $-2 + 2i$	-2	2	$-2 - 2i$

Solution to Exercise 1.11

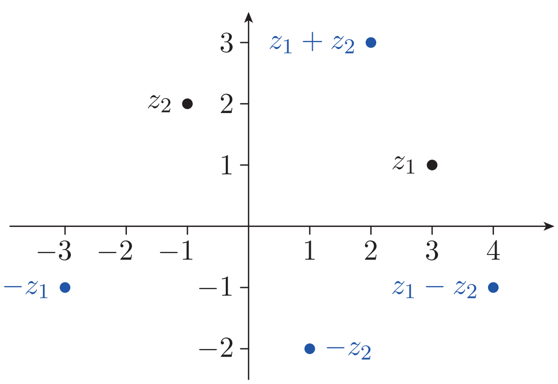
Let $z = x + iy$. Then $\overline{z} = x - iy$ and

$$\begin{aligned}\operatorname{Im}\overline{z} &= \operatorname{Im}(x - iy)\\ &= -y\\ &= -\operatorname{Im}(x+iy)\\ &= -\operatorname{Im} z.\end{aligned}$$

Solution to Exercise 2.1

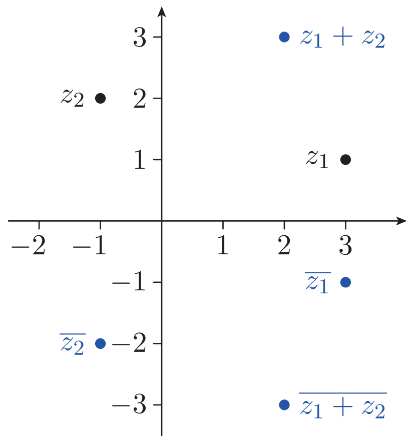
(a) With $z_1 = 3 + i$, $z_2 = -1 + 2i$, we have

$$-z_1 = -3 - i, \quad -z_2 = 1 - 2i,$$
$$z_1 + z_2 = 2 + 3i, \quad z_1 - z_2 = 4 - i.$$



(b) With $z_1 = 3 + i$, $z_2 = -1 + 2i$, we have

$$\overline{z_1} = 3 - i, \quad \overline{z_2} = -1 - 2i,$$
$$z_1 + z_2 = 2 + 3i, \quad \overline{z_1 + z_2} = 2 - 3i.$$



Solution to Exercise 2.2

(a) **(i)** $|1 + i| = \sqrt{1^2 + 1^2} = \sqrt{2}$

(ii) $|2 - 4i| = \sqrt{2^2 + (-4)^2} = \sqrt{20} = 2\sqrt{5}$

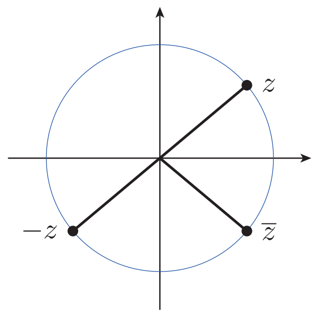
(iii) $|i| = \sqrt{1^2} = 1$

(iv) $|-5 + 12i| = \sqrt{(-5)^2 + 12^2} = \sqrt{169} = 13$

(b) Let $z = x + iy$. Then $\overline{z} = x - iy$ and $-z = -x - iy$. Hence

$$|\overline{z}| = \sqrt{x^2 + (-y)^2} = \sqrt{x^2 + y^2} = |z|,$$
$$|-z| = \sqrt{(-x)^2 + (-y)^2} = \sqrt{x^2 + y^2} = |z|.$$

Note that these results are 'obvious' geometrically.



Solution to Exercise 2.3

(a) Since $z_1 - z_2 = 4 - i$,

$$|z_1 - z_2| = \sqrt{4^2 + (-1)^2} = \sqrt{17}.$$

(b) Since $z_1 + z_2 = 2 + 3i$,

$$|z_1 + z_2| = \sqrt{2^2 + 3^2} = \sqrt{13}.$$

(c) The distance from z_2 to $-z_1$ is

$$|(-z_1) - z_2| = |z_1 + z_2| = \sqrt{13}.$$

Solution to Exercise 2.4

(a) Here $r = |i| = 1$, and the obvious choice for an argument of i is $\theta = \pi/2$. Thus

$$\begin{aligned} i &= 1(\cos\pi/2 + i\sin\pi/2) \\ &= \cos\pi/2 + i\sin\pi/2. \end{aligned}$$

(b) **(i)** $2(\cos\pi/3 + i\sin\pi/3)$

$$\begin{aligned} &= 2(1/2 + i\sqrt{3}/2) \\ &= 1 + \sqrt{3}i \end{aligned}$$

(ii) $3(\cos(-\pi/4) + i\sin(-\pi/4))$

$$\begin{aligned} &= 3(1/\sqrt{2} + i(-1/\sqrt{2})) \\ &= \frac{3}{\sqrt{2}} - \frac{3}{\sqrt{2}}i \end{aligned}$$

Solution to Exercise 2.5

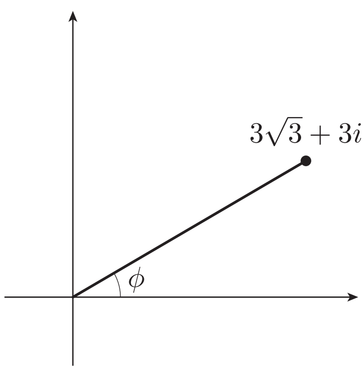
In each case we use the strategy for determining principal arguments to find $\operatorname{Arg} z$.

(a) -4 lies on the negative real axis, so $\operatorname{Arg}(-4) = \pi$ (see Figure 2.12). Since $|-4| = 4$, a polar form of -4 is

$$4(\cos\pi + i\sin\pi).$$

(b) $3\sqrt{3} + 3i$ lies in the upper-right quadrant, and

$$\phi = \tan^{-1}\frac{3}{3\sqrt{3}} = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}.$$



Thus the principal argument θ is

$$\begin{aligned} \theta &= \phi \quad \text{(see Figure 2.13(b))} \\ &= \pi/6. \end{aligned}$$

Since

$$|3\sqrt{3} + 3i| = \sqrt{(3\sqrt{3})^2 + 3^2} = \sqrt{27+9} = 6,$$

a polar form of $3\sqrt{3} + 3i$ is

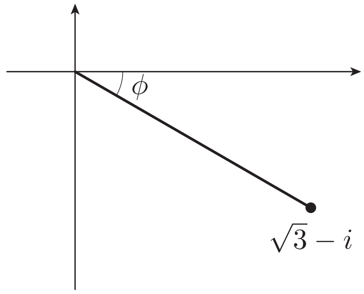
$$6(\cos\pi/6 + i\sin\pi/6).$$

(An alternative way to calculate the modulus is to observe that

$$\begin{aligned} |3\sqrt{3} + 3i| &= |3||\sqrt{3} + i| \\ &= 3\sqrt{(\sqrt{3})^2 + 1^2} \\ &= 3 \times 2 = 6.) \end{aligned}$$

(c) $\sqrt{3} - i$ lies in the lower-right quadrant, and

$$\phi = \tan^{-1}\frac{|-1|}{\sqrt{3}} = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}.$$



Thus the principal argument θ is

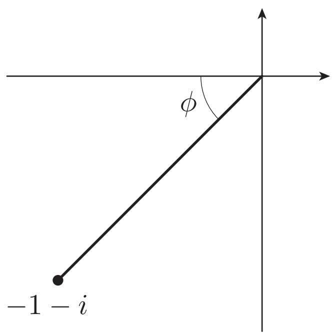
$$\begin{aligned} \theta &= -\phi \quad \text{(see Figure 2.13(b))} \\ &= -\pi/6. \end{aligned}$$

Since $|\sqrt{3} - i| = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2$, a polar form of $\sqrt{3} - i$ is

$$2(\cos(-\pi/6) + i\sin(-\pi/6)).$$

(d) $-1 - i$ lies in the lower-left quadrant, and

$$\phi = \tan^{-1}\frac{|-1|}{|-1|} = \tan^{-1} 1 = \frac{\pi}{4}.$$



Thus the principal argument θ is

$$\begin{aligned} \theta &= -(\pi - \phi) \quad \text{(see Figure 2.13(b))} \\ &= -(\pi - \pi/4) = -3\pi/4. \end{aligned}$$

Since $|-1 - i| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$, a polar form of $-1 - i$ is

$$\sqrt{2}(\cos(-3\pi/4) + i\sin(-3\pi/4)).$$

Solution to Exercise 2.6

$|z_1| = |-1-\sqrt{3}i| = 2$ and, from Example 2.2(b), an argument of z_1 is $-2\pi/3$; so

$$z_1 = 2(\cos(-2\pi/3) + i\sin(-2\pi/3)).$$

From Exercise 2.5(b),

$$z_2 = 6(\cos\pi/6 + i\sin\pi/6).$$

Thus, from formula (2.1),

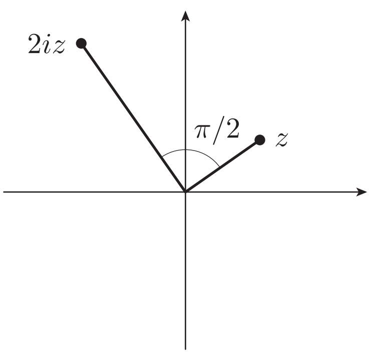
$$\begin{aligned} z_1 z_2 &= 2\times 6(\cos(-2\pi/3+\pi/6) \\ &\qquad\qquad + i\sin(-2\pi/3+\pi/6)) \\ &= 12(\cos(-\pi/2) + i\sin(-\pi/2)) \\ &= -12i \end{aligned}$$

and

$$\begin{aligned} z_1^2 &= z_1 z_1 \\ &= 2\times 2(\cos(-2\pi/3 - 2\pi/3) \\ &\qquad\qquad + i\sin(-2\pi/3 - 2\pi/3)) \\ &= 4(\cos(-4\pi/3) + i\sin(-4\pi/3)) \\ &= 4(-1/2 + i\sqrt{3}/2) \\ &= -2 + 2\sqrt{3}i. \end{aligned}$$

Solution to Exercise 2.7

Since $|2i| = 2$ and $\operatorname{Arg}(2i) = \pi/2$, multiplying z by $2i$ scales z by the factor 2 and rotates it anticlockwise through $\pi/2$ about 0.



Solution to Exercise 2.8

A polar form of $1+\sqrt{3}i$ is

$$z_1 = 2(\cos\pi/3 + i\sin\pi/3)$$

and, from Exercise 2.5(c), a polar form of $\sqrt{3}-i$ is

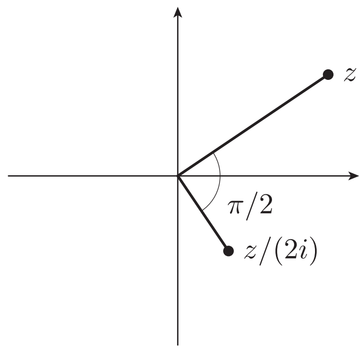
$$z_2 = 2(\cos(-\pi/6) + i\sin(-\pi/6)).$$

Thus, from formula (2.2),

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{2}{2}(\cos(\pi/3 - (-\pi/6)) \\ &\qquad + i\sin(\pi/3 - (-\pi/6))) \\ &= \cos\pi/2 + i\sin\pi/2 \\ &= i. \end{aligned}$$

Solution to Exercise 2.9

Since $|2i| = 2$ and $\operatorname{Arg}(2i) = \pi/2$, dividing z by $2i$ scales z by the factor $\frac{1}{2}$ and rotates it clockwise through $\pi/2$ about 0.



Solution to Exercise 2.10

Since $1+i = \sqrt{2}(\cos\pi/4 + i\sin\pi/4)$, we have

$$\begin{aligned} (1+i)^{-1} &= \frac{1}{\sqrt{2}}(\cos(-\pi/4) + i\sin(-\pi/4)) \\ &= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) \\ &= \frac{1}{2} - \frac{1}{2}i. \end{aligned}$$

Solution to Exercise 2.11

Since

$$\begin{aligned} z_1 &= 1+i = \sqrt{2}(\cos\pi/4 + i\sin\pi/4), \\ z_2 &= 1+\sqrt{3}i = 2(\cos\pi/3 + i\sin\pi/3), \\ z_3 &= \sqrt{3}+i = 2(\cos\pi/6 + i\sin\pi/6), \end{aligned}$$

we have

$$\begin{aligned} z_1 z_2 z_3 &= \sqrt{2}\times 2\times 2(\cos(\pi/4+\pi/3+\pi/6) \\ &\qquad\qquad + i\sin(\pi/4+\pi/3+\pi/6)) \\ &= 4\sqrt{2}(\cos 3\pi/4 + i\sin 3\pi/4) \\ &= 4\sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) \\ &= -4+4i. \end{aligned}$$

Solution to Exercise 2.12

(a) Since

$$\sqrt{3}+i = 2(\cos\pi/6 + i\sin\pi/6),$$

De Moivre's Theorem gives

$$\begin{aligned}(\sqrt{3}+i)^4 &= 2^4(\cos 4\pi/6 + i\sin 4\pi/6)\\ &= 16(\cos 2\pi/3 + i\sin 2\pi/3)\\ &= 16\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\\ &= -8+8\sqrt{3}i.\end{aligned}$$

(b) Since

$$1-\sqrt{3}i = 2(\cos(-\pi/3) + i\sin(-\pi/3)),$$

De Moivre's Theorem gives

$$\begin{aligned}(1-\sqrt{3}i)^3 &= 2^3(\cos(-3\pi/3) + i\sin(-3\pi/3))\\ &= 8(\cos(-\pi) + i\sin(-\pi))\\ &= -8.\end{aligned}$$

(c) Since

$$1+i = \sqrt{2}(\cos\pi/4 + i\sin\pi/4),$$

De Moivre's Theorem gives

$$\begin{aligned}(1+i)^{10} &= (\sqrt{2})^{10}(\cos 10\pi/4 + i\sin 10\pi/4)\\ &= 2^5(\cos\pi/2 + i\sin\pi/2)\\ &= 32i.\end{aligned}$$

(d) Since

$$-1+i = \sqrt{2}(\cos 3\pi/4 + i\sin 3\pi/4),$$

De Moivre's Theorem gives

$$\begin{aligned}&(-1+i)^{-8}\\ &= (\sqrt{2})^{-8}(\cos(-24\pi/4) + i\sin(-24\pi/4))\\ &= 2^{-4}(\cos(-6\pi) + i\sin(-6\pi))\\ &= \tfrac{1}{16}.\end{aligned}$$

(e) Since

$$\sqrt{3}+i = 2(\cos\pi/6 + i\sin\pi/6),$$

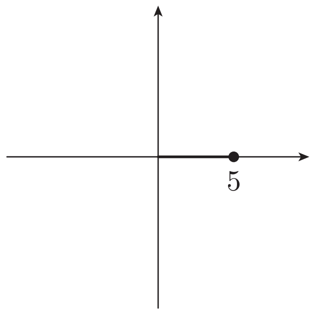
De Moivre's Theorem gives

$$\begin{aligned}&(\sqrt{3}+i)^{-6}\\ &= 2^{-6}(\cos(-6\pi/6) + i\sin(-6\pi/6))\\ &= 2^{-6}(\cos(-\pi) + i\sin(-\pi))\\ &= -\tfrac{1}{64}.\end{aligned}$$

Solution to Exercise 2.13

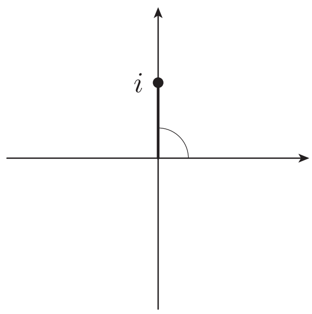
If you can 'see' what the principal argument is from your diagram, then write it down. Our calculations use the strategy for determining principal arguments from Subsection 2.2.

(a)



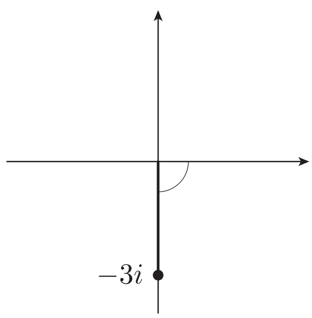
$5 = 5(\cos 0 + i\sin 0)$

(b)



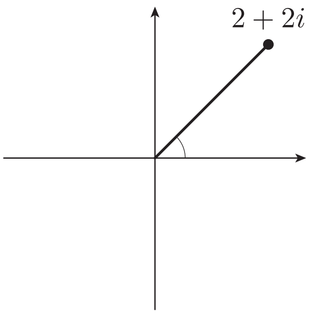
$$i = 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

(c)



$$-3i = 3\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$$

(d)



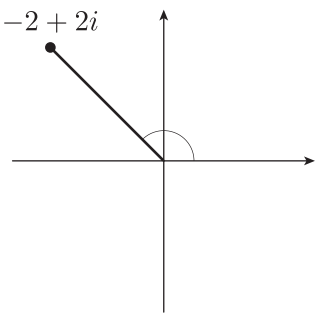
$$|2+2i| = \sqrt{8} = 2\sqrt{2},$$

$$\operatorname{Arg}(2+2i) = \tan^{-1}\frac{2}{2} = \frac{\pi}{4},$$

so

$$2+2i = 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right).$$

(e)



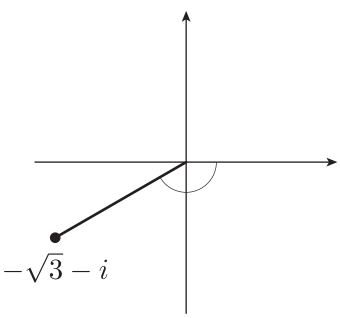
$$|-2+2i| = \sqrt{8} = 2\sqrt{2},$$

$$\operatorname{Arg}(-2+2i) = \pi - \tan^{-1}\frac{2}{2} = \frac{3\pi}{4},$$

so

$$-2+2i = 2\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right).$$

(f)



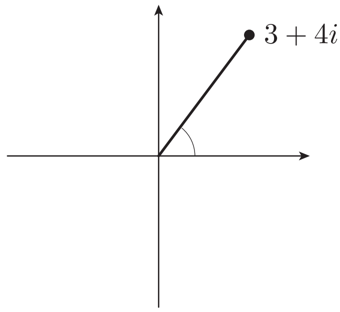
$$|-\sqrt{3}-i| = 2,$$

$$\operatorname{Arg}(-\sqrt{3}-i) = -\left(\pi - \tan^{-1}\frac{1}{\sqrt{3}}\right) = -\frac{5\pi}{6},$$

so

$$-\sqrt{3}-i = 2\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right).$$

(g)

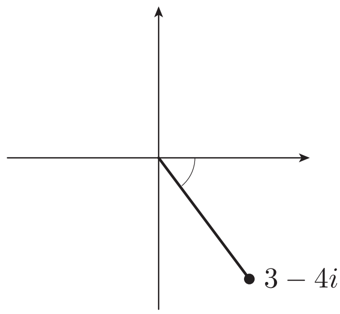


$|3+4i| = \sqrt{25} = 5,\ \operatorname{Arg}(3+4i) = \tan^{-1}\frac{4}{3}$, so

$$3+4i = 5(\cos\theta + i\sin\theta),$$

where $\theta = \tan^{-1}\frac{4}{3} \approx 0.927\,\text{rad}$.

(h)



$|3-4i| = \sqrt{25} = 5,\ \operatorname{Arg}(3-4i) = -\tan^{-1}\frac{4}{3}$, so

$$3-4i = 5(\cos\theta + i\sin\theta),$$

where $\theta = -\tan^{-1}\frac{4}{3} \approx -0.927\,\text{rad}$.

Alternatively, part (g) tells us that

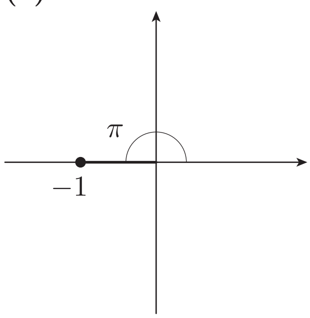
$$3+4i = 5(\cos\theta + i\sin\theta),$$

where $\theta = \tan^{-1}\frac{4}{3}$, so

$$3-4i = \overline{3+4i} = 5(\cos(-\theta) + i\sin(-\theta)).$$

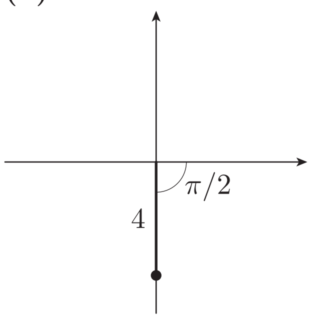
Solution to Exercise 2.14

(a)



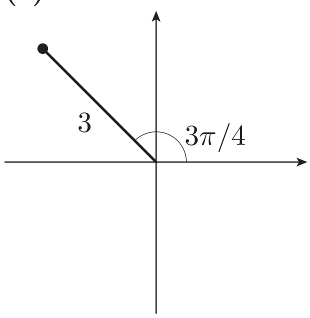
$\cos\pi + i\sin\pi = -1$

(b)



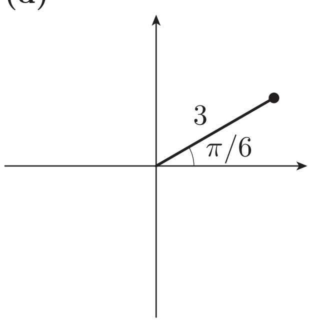
$4\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right) = -4i$

(c)



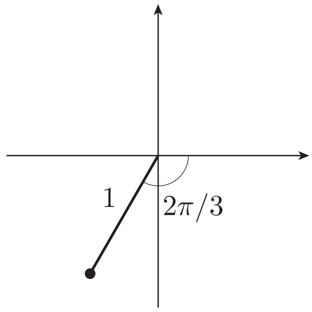
$3\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = -\frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}}i$

(d)



$3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{2} + \frac{3}{2}i$

(e)



$\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

Solution to Exercise 2.15

(a) $$\begin{aligned}|z_2 - z_1| &= |(2+3i) - (1+i)|\\ &= |1+2i|\\ &= \sqrt{1+4} = \sqrt{5}\end{aligned}$$

(b) $$\begin{aligned}|z_2 - z_1| &= |(1-7i) - (-2+3i)|\\ &= |3-10i|\\ &= \sqrt{9+100} = \sqrt{109}\end{aligned}$$

(c) $$\begin{aligned}|z_2 - z_1| &= |-i-i|\\ &= |-2i|\\ &= 2\end{aligned}$$

Solution to Exercise 2.16

(a) Since

$$1 + \sqrt{3}i = 2(\cos\pi/3 + i\sin\pi/3),$$

De Moivre's Theorem gives

$$\begin{aligned}(1+\sqrt{3}i)^5 &= (2(\cos\pi/3 + i\sin\pi/3))^5\\ &= 2^5(\cos 5\pi/3 + i\sin 5\pi/3)\\ &= 32\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\\ &= 16 - 16\sqrt{3}i.\end{aligned}$$

(b) Since

$$1 + i = \sqrt{2}(\cos\pi/4 + i\sin\pi/4),$$

De Moivre's Theorem gives

$$\begin{aligned}(1+i)^{-4} &= \left(\sqrt{2}(\cos\pi/4 + i\sin\pi/4)\right)^{-4}\\ &= 2^{-4/2}(\cos(-4\pi/4) + i\sin(-4\pi/4))\\ &= \tfrac{1}{4}(\cos(-\pi) + i\sin(-\pi))\\ &= -\tfrac{1}{4}.\end{aligned}$$

(c) Since

$$1 + i = \sqrt{2}(\cos \pi/4 + i \sin \pi/4),$$

De Moivre's Theorem gives

$$\begin{aligned}(1+i)^6 &= \left(\sqrt{2}(\cos \pi/4 + i \sin \pi/4)\right)^6 \\ &= 2^3(\cos 3\pi/2 + i \sin 3\pi/2).\end{aligned}$$

Also,

$$\sqrt{3} - i = 2(\cos(-\pi/6) + i \sin(-\pi/6)),$$

so De Moivre's Theorem gives

$$\begin{aligned}(\sqrt{3} - i)^{-3} &= (2(\cos(-\pi/6) + i \sin(-\pi/6)))^{-3} \\ &= 2^{-3}(\cos \pi/2 + i \sin \pi/2).\end{aligned}$$

Hence

$$\begin{aligned}\frac{(1+i)^6}{(\sqrt{3}-i)^3} &= 2^3(\cos 3\pi/2 + i \sin 3\pi/2) \\ &\quad \times 2^{-3}(\cos \pi/2 + i \sin \pi/2) \\ &= \cos 2\pi + i \sin 2\pi = 1.\end{aligned}$$

Solution to Exercise 2.17

By De Moivre's Theorem,

$$(\cos\theta + i \sin\theta)^3 = \cos 3\theta + i \sin 3\theta.$$

By the Binomial Theorem,

$$\begin{aligned}(\cos\theta + i\sin\theta)^3 &= \cos^3\theta + 3\cos^2\theta(i\sin\theta) \\ &\quad + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3 \\ &= \cos^3\theta - 3\cos\theta\sin^2\theta \\ &\quad + i(3\cos^2\theta\sin\theta - \sin^3\theta).\end{aligned}$$

The two expressions we have obtained for $(\cos\theta + i\sin\theta)^3$ are equal, so their real parts are equal and their imaginary parts are equal. Equating the two imaginary parts gives

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta.$$

Since $\cos^2\theta = 1 - \sin^2\theta$, we have

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta,$$

as required.

Solution to Exercise 2.18

Since $(x + iy)^4 = a + ib$, it follows that

$$|x + iy|^4 = |(x+iy)^4| = |a + ib|.$$

Squaring the left-hand side gives

$$(|x+iy|^4)^2 = (|x+iy|^2)^4 = (x^2+y^2)^4,$$

and squaring the right-hand side gives $a^2 + b^2$. Hence

$$(x^2+y^2)^4 = a^2 + b^2.$$

Solution to Exercise 2.19

If $\overline{z} = z^{-1}$, then

$$z\overline{z} = zz^{-1} = 1.$$

Hence, by Theorem 2.1(c),

$$|z|^2 = z\overline{z} = 1,$$

so $|z| = 1$, as required.

Solution to Exercise 3.1

Since $-1 + \sqrt{3}i = 2(\cos 2\pi/3 + i \sin 2\pi/3)$, a solution of $z^2 = -1 + \sqrt{3}i$ is obtained by taking z to have modulus $\sqrt{2}$ and argument $\frac{1}{2}(2\pi/3) = \pi/3$.

This gives

$$\begin{aligned}z &= \sqrt{2}(\cos \pi/3 + i \sin \pi/3) \\ &= \sqrt{2}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{\sqrt{2}}i.\end{aligned}$$

Therefore the required solutions are

$$z = \pm\left(\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{\sqrt{2}}i\right).$$

Solution to Exercise 3.2

(a) Using the principal argument of $8i$, we have

$$8i = 8(\cos \pi/2 + i \sin \pi/2),$$

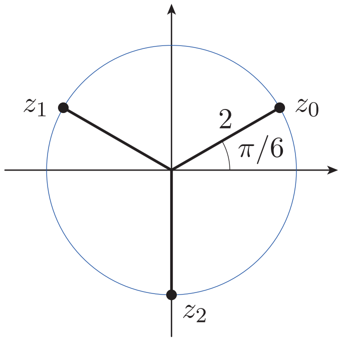
and, using the strategy for finding nth roots, we deduce that the cube roots of $8i$ are

$$z_k = 8^{1/3}\left(\cos\left(\frac{\pi}{6} + k\frac{2\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + k\frac{2\pi}{3}\right)\right),$$

where $k = 0, 1, 2$; that is,

$$\begin{aligned}z_0 &= 2(\cos \pi/6 + i \sin \pi/6) = \sqrt{3} + i, \\ z_1 &= 2(\cos 5\pi/6 + i \sin 5\pi/6) = -\sqrt{3} + i, \\ z_2 &= 2(\cos 3\pi/2 + i \sin 3\pi/2) = -2i.\end{aligned}$$

Since the principal argument of $8i$ is $\pi/2$, the principal cube root of $8i$ is z_0.



(b) The principal argument of $-i$ is $-\pi/2$; to avoid negative signs, we use the argument $3\pi/2$. Thus

$$-i = \cos 3\pi/2 + i \sin 3\pi/2,$$

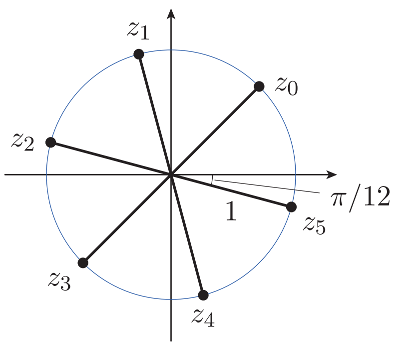
and, using the strategy for finding nth roots, we deduce that the sixth roots of $-i$ are

$$\begin{aligned} z_k &= \cos\left(\frac{3\pi/2}{6} + k\frac{2\pi}{6}\right) + i\sin\left(\frac{3\pi/2}{6} + k\frac{2\pi}{6}\right) \\ &= \cos\left(\frac{\pi}{4} + k\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{4} + k\frac{\pi}{3}\right), \end{aligned}$$

where $k = 0, 1, \ldots, 5$; that is,

$$\begin{aligned} z_0 &= \cos \pi/4 + i \sin \pi/4, \\ z_1 &= \cos 7\pi/12 + i \sin 7\pi/12, \\ z_2 &= \cos 11\pi/12 + i \sin 11\pi/12, \\ z_3 &= \cos 5\pi/4 + i \sin 5\pi/4, \\ z_4 &= \cos 19\pi/12 + i \sin 19\pi/12, \\ z_5 &= \cos 23\pi/12 + i \sin 23\pi/12. \end{aligned}$$

Since the principal argument of $-i$ is $-\pi/2$, the principal sixth root of $-i$ has argument $-\pi/12$ and hence it is z_5.



Alternatively, using the principal argument, we have

$$-i = \cos(-\pi/2) + i \sin(-\pi/2),$$

and, using the strategy, we deduce that the sixth roots of $-i$ are

$$z_k = \cos\left(-\frac{\pi}{12} + k\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{12} + k\frac{\pi}{3}\right),$$

where $k = 0, 1, \ldots, 5$; that is,

$$\begin{aligned} z_0 &= \cos(-\pi/12) + i \sin(-\pi/12), \\ z_1 &= \cos \pi/4 + i \sin \pi/4, \\ z_2 &= \cos 7\pi/12 + i \sin 7\pi/12, \\ z_3 &= \cos 11\pi/12 + i \sin 11\pi/2, \\ z_4 &= \cos 5\pi/4 + i \sin 5\pi/4, \\ z_5 &= \cos 19\pi/12 + i \sin 19\pi/12. \end{aligned}$$

In this case, the principal sixth root is

$$z_0 = \cos(-\pi/12) + i \sin(-\pi/12).$$

Solution to Exercise 3.3

(a) By the Geometric Series Identity,

$$1 - z^n = (1 - z)(1 + z + z^2 + \cdots + z^{n-1}).$$

Thus if

$$z^n = 1 \quad \text{and} \quad z \neq 1,$$

then

$$1 - z^n = 0 \quad \text{and} \quad 1 - z \neq 0,$$

so

$$1 + z + z^2 + \cdots + z^{n-1} = 0,$$

as required.

(b) By De Moivre's Theorem and the corollary to Theorem 3.1, the nth roots of unity are of the form

$$1, z, z^2, \ldots, z^{n-1},$$

where

$$z = \cos 2\pi/n + i \sin 2\pi/n.$$

Hence, by part (a), the sum of the nth roots of unity is 0.

Solution to Exercise 3.4

(a) The factorisation

$$z^2 - 7iz + 8 = (z+i)(z-8i) = 0$$

shows that the solutions are $z = -i, 8i$.

(b) Formula (3.1) with $a = 1$, $b = 2$, $c = 1 - i$ gives

$$\begin{aligned} z &= \frac{-2 \pm \sqrt{4 - 4(1-i)}}{2} \\ &= -1 \pm \sqrt{i} \\ &= -1 \pm \frac{1}{\sqrt{2}}(1+i) \quad \text{(by Example 3.1).} \end{aligned}$$

So the solutions are

$$z = \left(-1 + \frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}i, \left(-1 - \frac{1}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}}i.$$

Solution to Exercise 3.5

(a) Putting $w = z^3$ gives

$$w^2 - 7iw + 8 = 0,$$

so $w = -i$ or $w = 8i$ (see Exercise 3.4(a)). Thus the solutions are the cube roots of $-i$ and $8i$. We found the cube roots of $8i$ in Exercise 3.2(a); these are

$$\sqrt{3} + i, \quad -\sqrt{3} + i \quad \text{and} \quad -2i.$$

By a similar method, or using the hint, we find that the cube roots of $-i$ are

$$-\frac{\sqrt{3}}{2} - \frac{1}{2}i, \quad \frac{\sqrt{3}}{2} - \frac{1}{2}i \quad \text{and} \quad i.$$

Hence the six solutions are

$$\begin{aligned} &\sqrt{3} + i,\ -\sqrt{3} + i,\ -2i, \\ &-\frac{\sqrt{3}}{2} - \frac{1}{2}i,\ \frac{\sqrt{3}}{2} - \frac{1}{2}i,\ i. \end{aligned}$$

(b) Putting $w = z^2$ gives

$$w^2 + 4iw + 8 = 0,$$

so

$$w = \frac{-4i \pm \sqrt{-16 - 32}}{2};$$

that is,

$$w = (\sqrt{12} - 2)i \quad \text{or} \quad w = -(\sqrt{12} + 2)i.$$

Since

$$(\sqrt{12} - 2)i = (\sqrt{12} - 2)(\cos \pi/2 + i \sin \pi/2)$$

and

$$-(\sqrt{12} + 2)i = (\sqrt{12} + 2)(\cos 3\pi/2 + i \sin 3\pi/2),$$

the four solutions are

$$\begin{aligned} z &= \pm(\sqrt{12} - 2)^{1/2}(\cos \pi/4 + i \sin \pi/4) \\ &= \pm(\sqrt{12} - 2)^{1/2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) \\ &= \pm(\sqrt{3} - 1)^{1/2}(1+i) \end{aligned}$$

and

$$\begin{aligned} z &= \pm(\sqrt{12} + 2)^{1/2}(\cos 3\pi/4 + i \sin 3\pi/4) \\ &= \pm(\sqrt{12} + 2)^{1/2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) \\ &= \pm(\sqrt{3} + 1)^{1/2}(-1+i). \end{aligned}$$

Solution to Exercise 3.6

In each case we will express the given complex number in polar form using the principal argument.

(a) **(i)** As $-i = \cos(-\pi/2) + i \sin(-\pi/2)$, the square roots of $-i$ are

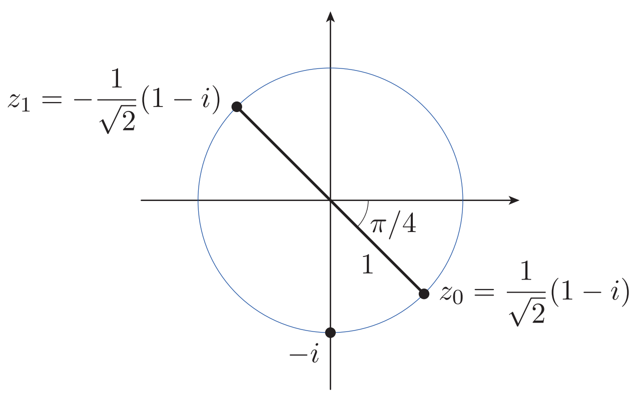
$$z_k = \cos(-\pi/4 + k\pi) + i \sin(-\pi/4 + k\pi),$$

$k = 0, 1$. Thus the principal square root is

$$z_0 = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i = \frac{1}{\sqrt{2}}(1-i),$$

and the other root is

$$z_1 = -z_0 = -\frac{1}{\sqrt{2}}(1-i).$$



(ii) As $4i = 4(\cos\pi/2 + i\sin\pi/2)$, the square roots of $4i$ are

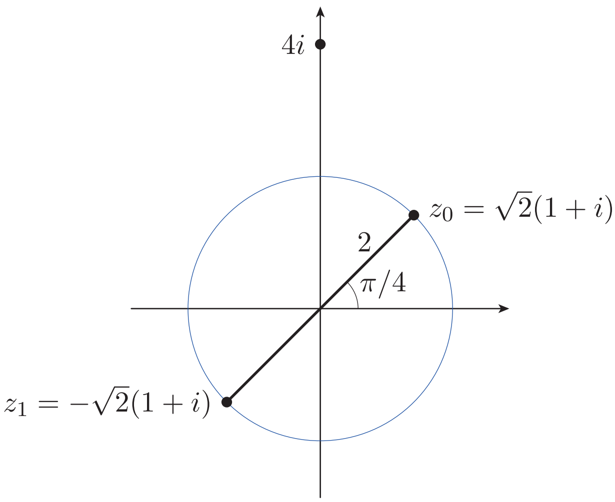
$$z_k = 2(\cos(\pi/4 + k\pi) + i\sin(\pi/4 + k\pi)),$$

$k = 0, 1$. Thus the principal square root is

$$z_0 = \sqrt{2} + \sqrt{2}i = \sqrt{2}(1+i),$$

and the other root is

$$z_1 = -z_0 = -\sqrt{2}(1+i).$$



(b) (i) As $-1 = \cos\pi + i\sin\pi$, the cube roots of -1 are

$$z_k = \cos\left(\frac{\pi}{3} + k\frac{2\pi}{3}\right) + i\sin\left(\frac{\pi}{3} + k\frac{2\pi}{3}\right),$$

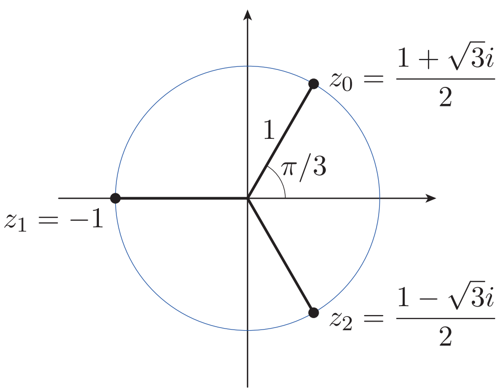
$k = 0, 1, 2$. The principal cube root is

$$z_0 = \cos\pi/3 + i\sin\pi/3 = \frac{1+\sqrt{3}i}{2},$$

and the other roots are

$$z_1 = \cos\pi + i\sin\pi = -1,$$

$$z_2 = \cos 5\pi/3 + i\sin 5\pi/3 = \frac{1-\sqrt{3}i}{2}.$$



(ii) As $-2 + 2i = 2\sqrt{2}(\cos 3\pi/4 + i\sin 3\pi/4)$, the cube roots of $-2+2i$ are

$$z_k = \sqrt{2}\left(\cos\left(\frac{\pi}{4} + k\frac{2\pi}{3}\right) + i\sin\left(\frac{\pi}{4} + k\frac{2\pi}{3}\right)\right),$$

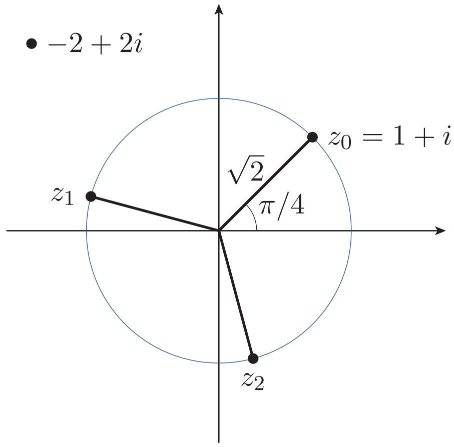
$k = 0, 1, 2$. The principal cube root is

$$z_0 = \sqrt{2}(\cos\pi/4 + i\sin\pi/4) = 1 + i,$$

and the other roots are

$$z_1 = \sqrt{2}(\cos 11\pi/12 + i\sin 11\pi/12),$$

$$z_2 = \sqrt{2}(\cos 19\pi/12 + i\sin 19\pi/12).$$



(c) (i) As

$$\frac{1}{\sqrt{2}}(-1-i) = \cos(-3\pi/4) + i\sin(-3\pi/4),$$

the fourth roots of $\frac{1}{\sqrt{2}}(-1-i)$ are

$$z_k = \cos\left(-\frac{3\pi}{16} + k\frac{\pi}{2}\right) + i\sin\left(-\frac{3\pi}{16} + k\frac{\pi}{2}\right),$$

$k = 0, 1, 2, 3$. The principal fourth root is

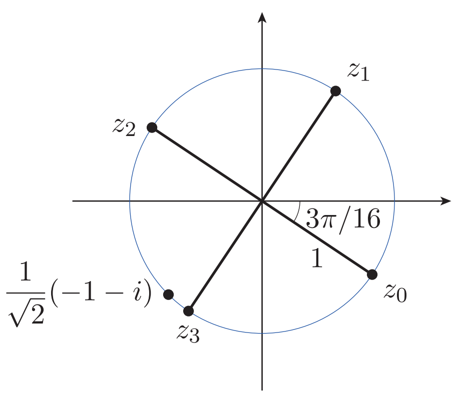
$$z_0 = \cos(-3\pi/16) + i\sin(-3\pi/16),$$

and the other roots are

$$z_1 = \cos 5\pi/16 + i\sin 5\pi/16,$$

$$z_2 = \cos 13\pi/16 + i\sin 13\pi/16,$$

$$z_3 = \cos 21\pi/16 + i\sin 21\pi/16.$$



(ii) As $-1+i=\sqrt{2}(\cos 3\pi/4+i\sin 3\pi/4)$, the fourth roots of $-1+i$ are

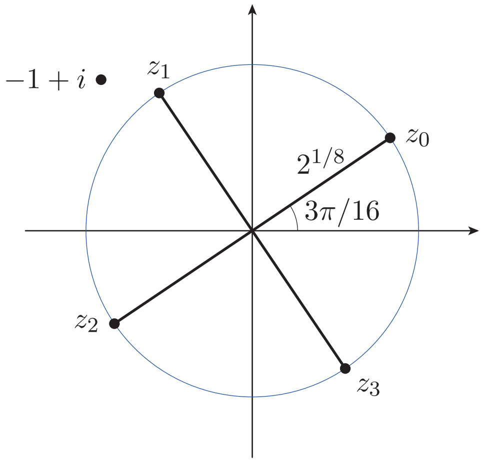
$$z_k = 2^{1/8}\left(\cos\left(\frac{3\pi}{16}+k\frac{\pi}{2}\right)+i\sin\left(\frac{3\pi}{16}+k\frac{\pi}{2}\right)\right),$$

$k=0,1,2,3$. The principal fourth root is

$$z_0 = 2^{1/8}(\cos 3\pi/16 + i\sin 3\pi/16),$$

and the other roots are

$$z_1 = 2^{1/8}(\cos 11\pi/16 + i\sin 11\pi/16),$$
$$z_2 = 2^{1/8}(\cos 19\pi/16 + i\sin 19\pi/16),$$
$$z_3 = 2^{1/8}(\cos 27\pi/16 + i\sin 27\pi/16).$$



(d) (i) As $-1=\cos\pi+i\sin\pi$, the fifth roots of -1 are

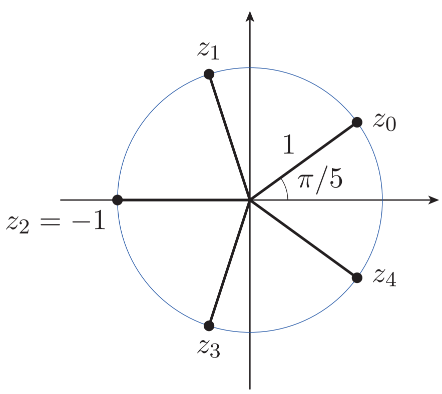
$$z_k = \cos\left(\frac{\pi}{5}+k\frac{2\pi}{5}\right)+i\sin\left(\frac{\pi}{5}+k\frac{2\pi}{5}\right),$$

$k=0,1,\ldots,4$. The principal fifth root is

$$z_0 = \cos\pi/5 + i\sin\pi/5,$$

and the other roots are

$$z_1 = \cos 3\pi/5 + i\sin 3\pi/5,$$
$$z_2 = \cos\pi + i\sin\pi = -1,$$
$$z_3 = \cos 7\pi/5 + i\sin 7\pi/5,$$
$$z_4 = \cos 9\pi/5 + i\sin 9\pi/5.$$



(ii) As $-16+16\sqrt{3}i = 32(\cos 2\pi/3 + i\sin 2\pi/3)$, the fifth roots of $-16+16\sqrt{3}i$ are

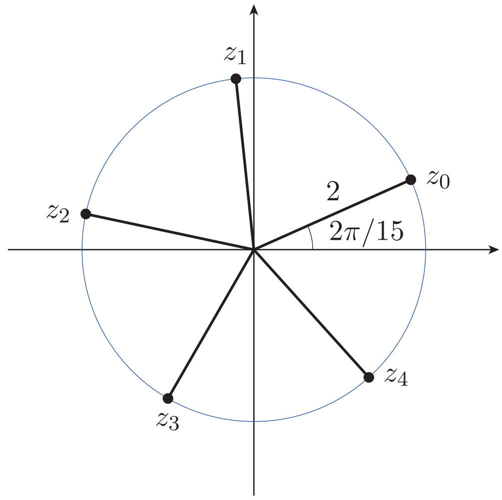
$$z_k = 2\left(\cos\left(\frac{2\pi}{15}+k\frac{2\pi}{5}\right)+i\sin\left(\frac{2\pi}{15}+k\frac{2\pi}{5}\right)\right),$$

$k=0,1,\ldots,4$. The principal fifth root is

$$z_0 = 2(\cos 2\pi/15 + i\sin 2\pi/15),$$

and the other roots are

$$z_1 = 2(\cos 8\pi/15 + i\sin 8\pi/15),$$
$$z_2 = 2(\cos 14\pi/15 + i\sin 14\pi/15),$$
$$z_3 = 2(\cos 4\pi/3 + i\sin 4\pi/3),$$
$$z_4 = 2(\cos 26\pi/15 + i\sin 26\pi/15).$$



Solution to Exercise 3.7

(a) Since $(x+iy)^2 = x^2-y^2+2xyi$, equating the real parts and imaginary parts of

$$(x+iy)^2 = 3+4i$$

gives

$$x^2-y^2=3, \quad 2xy=4. \tag{S1}$$

From the second equation, $y=2/x$, and substituting this in the first equation, we obtain

$$x^2-\frac{4}{x^2}=3;$$

that is,

$$x^4-3x^2-4=0$$

or

$$(x^2-4)(x^2+1)=0.$$

Since $x^2+1\neq 0$, it follows that $x^2=4$, so $x=\pm 2$. By equations (S1), when $x=2$ we have $y=1$, and when $x=-2$ we have $y=-1$. The two solutions are therefore

$$x+iy=\pm(2+i).$$

(b) Since $(x+iy)^2 = x^2 - y^2 + 2xyi$, equating the real parts and imaginary parts of

$$(x+iy)^2 = -5 + 12i$$

gives

$$x^2 - y^2 = -5, \quad 2xy = 12. \tag{S2}$$

From the second equation, $y = 6/x$, and substituting this in the first equation, we obtain

$$x^2 - \frac{36}{x^2} = -5;$$

that is,

$$x^4 + 5x^2 - 36 = 0$$

or

$$(x^2+9)(x^2-4) = 0.$$

Since $x^2 + 9 \neq 0$, it follows that $x^2 = 4$, so $x = \pm 2$. By equations (S2), when $x = 2$ we have $y = 3$, and when $x = -2$ we have $y = -3$. The two solutions are therefore

$$x + iy = \pm(2+3i).$$

Solution to Exercise 3.8

(a) Substituting $w = z^2$ in $z^4 - z^2 + 1 + i = 0$ gives

$$w^2 - w + 1 + i = 0,$$

which has solutions

$$\begin{aligned} w &= \frac{1 \pm \sqrt{1 - 4(1+i)}}{2} \\ &= \frac{1}{2} \pm \frac{1}{2}\sqrt{-(3+4i)} = \frac{1}{2} \pm \frac{i}{2}\sqrt{3+4i}. \end{aligned}$$

From Exercise 3.7(a), $\sqrt{3+4i} = 2+i$, and hence

$$w = \frac{1}{2} \pm \frac{i}{2}(2+i),$$

so $w = i$ or $w = 1 - i$.

(You may have spotted the factorisation

$$w^2 - w + 1 + i = (w-i)(w-1+i).)$$

Thus $z = \pm\sqrt{i}$ or $z = \pm\sqrt{1-i}$. Since

$$i = \cos\pi/2 + i\sin\pi/2$$

and

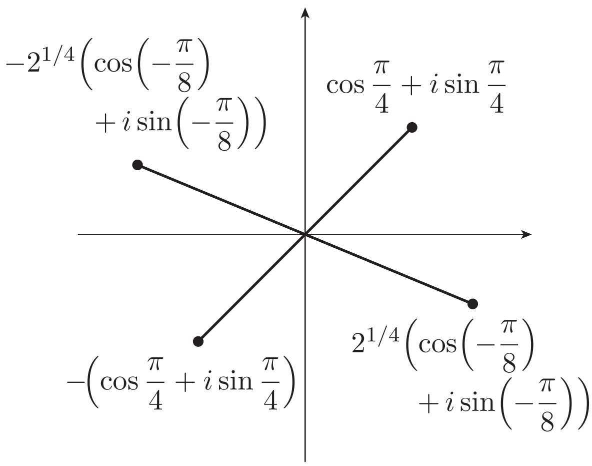
$$1 - i = \sqrt{2}(\cos(-\pi/4) + i\sin(-\pi/4)),$$

we obtain the four solutions

$$z = \pm(\cos\pi/4 + i\sin\pi/4)$$

and

$$z = \pm 2^{1/4}(\cos(-\pi/8) + i\sin(-\pi/8)).$$



(b) Since the polynomial $z^3 - 4z^2 + 6z - 4$ has real coefficients and is of odd degree, we expect there to be at least one real solution (because the graph of the real function $f(x) = x^3 - 4x^2 + 6x - 4$ crosses the x-axis). By trial and error (trying factors of 4), we find that $z = 2$ is a solution and hence

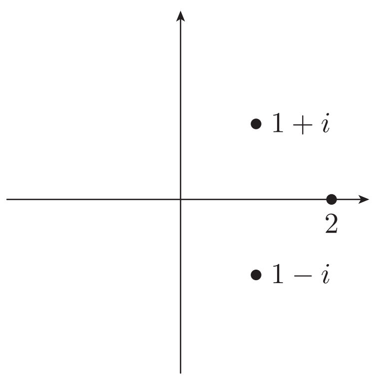
$$z^3 - 4z^2 + 6z - 4 = (z-2)(z^2 - 2z + 2).$$

Since the solutions of $z^2 - 2z + 2 = 0$ are

$$\begin{aligned} z &= \frac{2 \pm \sqrt{4 - 4 \times 2}}{2} \\ &= \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i, \end{aligned}$$

the solutions of the original equation are

$$z = 2,\ 1+i,\ 1-i.$$



Solution to Exercise 3.9

If z satisfies $p(z) = 0$, then

$$a_n z^n + \cdots + a_1 z + a_0 = 0.$$

Taking complex conjugates of both sides and using Theorem 1.1(b) gives

$$\begin{aligned} 0 &= \overline{a_n z^n + \cdots + a_1 z + a_0} \\ &= \overline{a_n z^n} + \cdots + \overline{a_1 z} + \overline{a_0} \\ &= a_n \overline{z}^n + \cdots + a_1 \overline{z} + a_0, \end{aligned}$$

since $a_0, a_1, \ldots, a_n$ are real. Hence $p(\overline{z}) = 0$, as required.

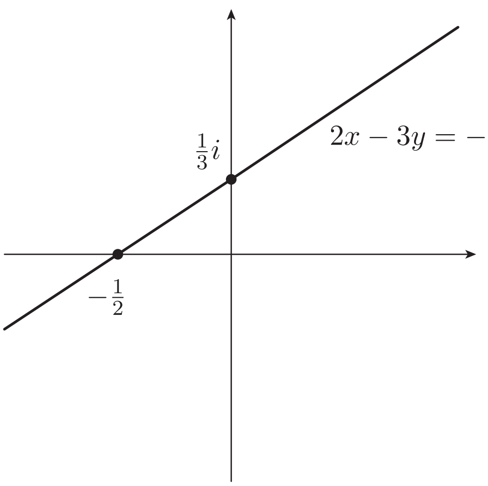
Remark: The solution to Exercise 3.8(b) provides an example of this result, whereas Exercise 3.8(a) shows that if the coefficients of a polynomial p are not all real, then solutions of $p(z) = 0$ need not occur in complex conjugate pairs.

Solution to Exercise 4.1

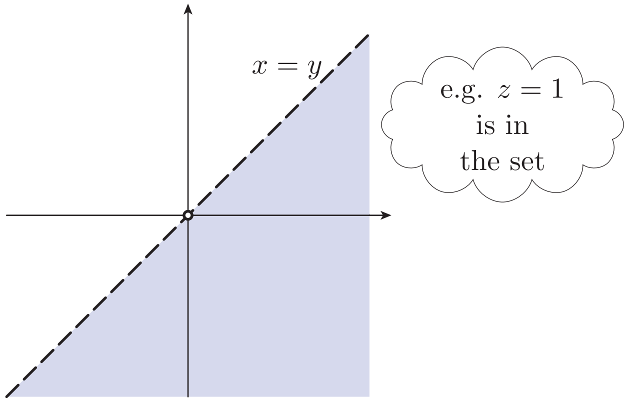
	$1+2i$	$-1-2i$	i	-2
$\operatorname{Re} z < 0$	×	✓	×	✓
$\lvert z\rvert > 2$	✓	✓	×	×
$\operatorname{Im} z \leq -1$	×	✓	×	×
$\operatorname{Arg} z \geq 0$	✓	×	✓	✓

Solution to Exercise 4.2

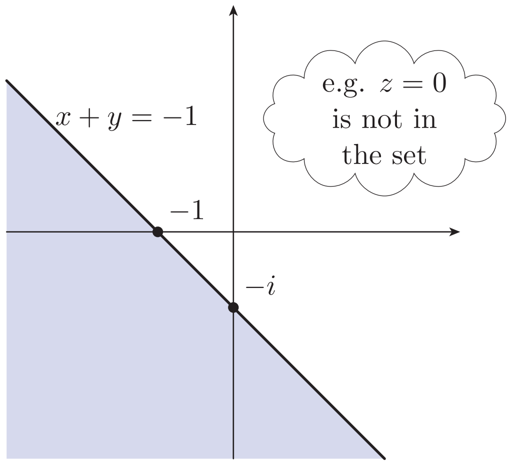
(a) (i) $\{z : 2\operatorname{Re} z - 3\operatorname{Im} z = -1\}$



(ii) $\{z : \operatorname{Re} z - \operatorname{Im} z > 0\}$



(iii) $\{z : \operatorname{Re} z + \operatorname{Im} z \leq -1\}$



(b) The half-plane includes its boundary, which has equation

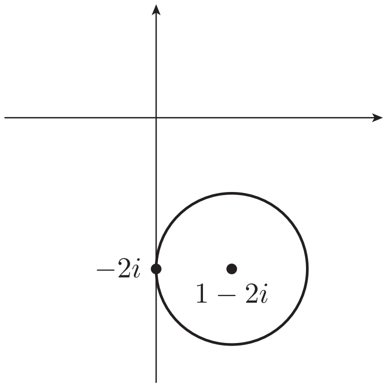
$$\tfrac{1}{2}x - y = -1.$$

At $z = 0$, $\frac{1}{2}x - y > -1$, and since 0 is not in this half-plane, the half-plane is the set

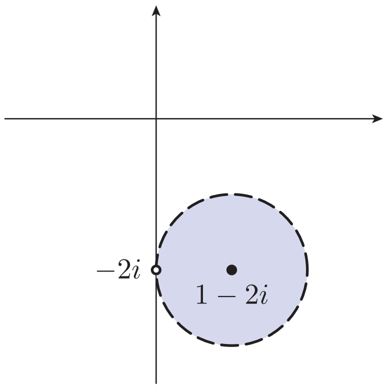
$$\left\{z : \tfrac{1}{2}\operatorname{Re} z - \operatorname{Im} z \leq -1\right\}.$$

Solution to Exercise 4.3

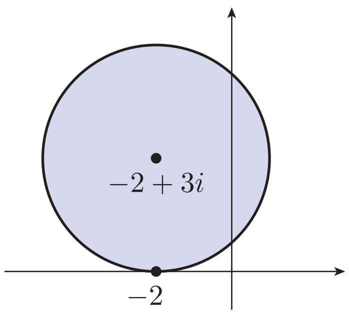
(a) **(i)** $\{z : |z - 1 + 2i| = 1\}$



(ii) $\{z : |z - 1 + 2i| < 1\}$



(iii) $\{z : |z + 2 - 3i| \leq 3\}$

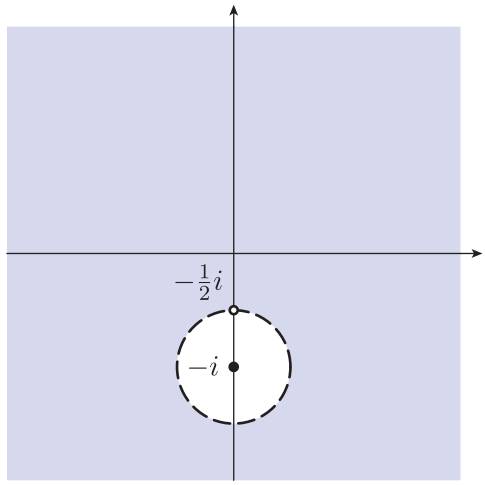


(b) This set is an open disc with centre $-1 - i$ and radius $|-1 - i| = \sqrt{2}$, so it is the set

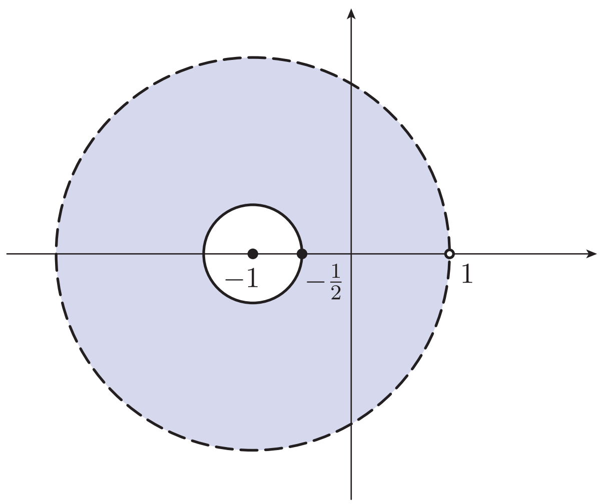
$$\{z : |z + 1 + i| < \sqrt{2}\}.$$

Solution to Exercise 4.4

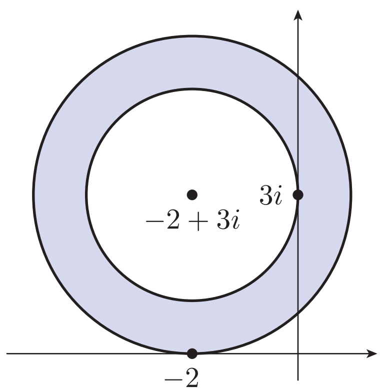
(a) **(i)** $\{z : |z + i| > \frac{1}{2}\}$



(ii) $\{z : \frac{1}{2} \leq |z + 1| < 2\}$



(iii) $\{z : 2 \leq |z + 2 - 3i| \leq 3\}$

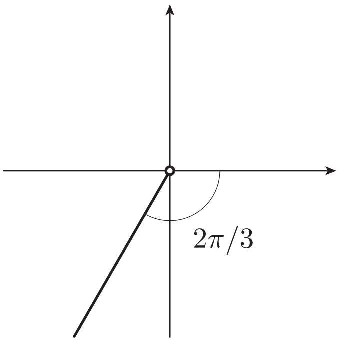


(b) This set is a punctured open disc with centre $1 - 2i$ and radius 1, so it is the set

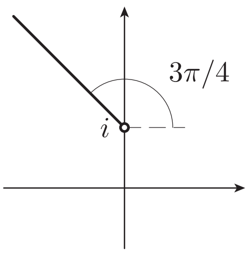
$$\{z : 0 < |z - 1 + 2i| < 1\}.$$

Solution to Exercise 4.5

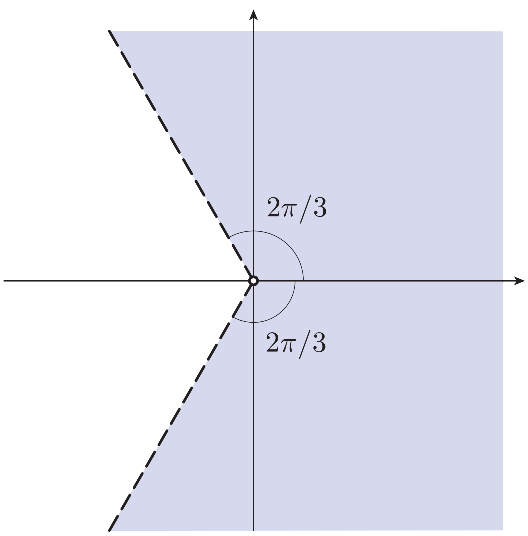
(a) **(i)** $\{z : \operatorname{Arg} z = -2\pi/3\}$



(ii) $\{z : \operatorname{Arg}(z - i) = 3\pi/4\}$



(iii) $\{z : |\operatorname{Arg} z| < 2\pi/3\}$

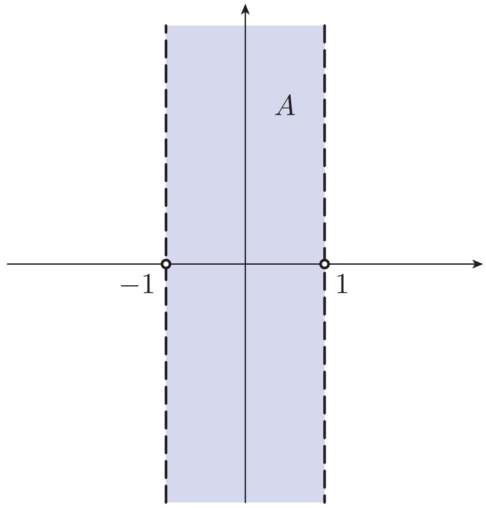


(b) This set is a sector (not an open one) with boundary rays $\{z : \operatorname{Arg}(z + 2i) = 0\}$ and $\{z : \operatorname{Arg}(z + 2i) = \pi/4\}$, so it is

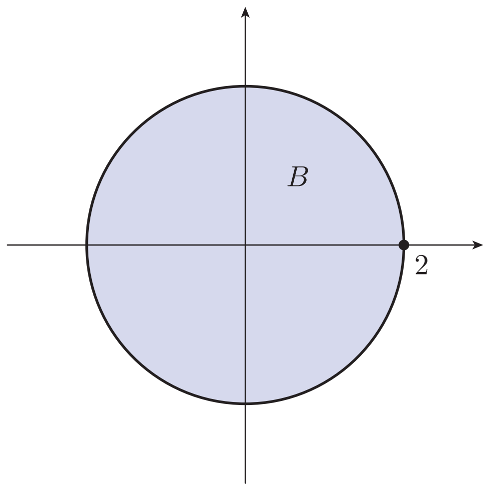
$$\{z : 0 \le \operatorname{Arg}(z + 2i) \le \pi/4\}.$$

Solution to Exercise 4.6

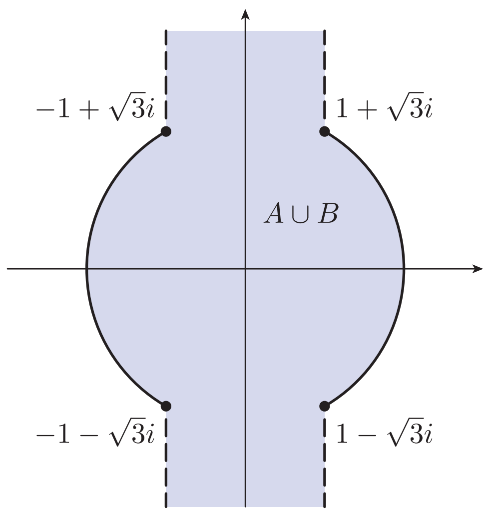
(a) **(i)** $A = \{z : |\operatorname{Re} z| < 1\}$



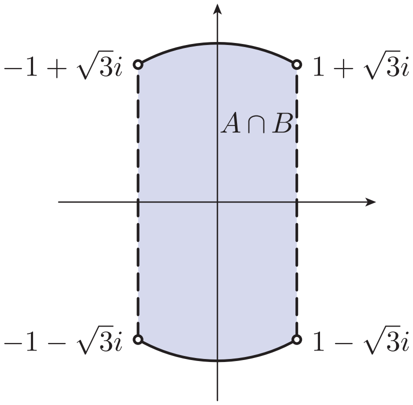
(ii) $B = \{z : |z| \le 2\}$



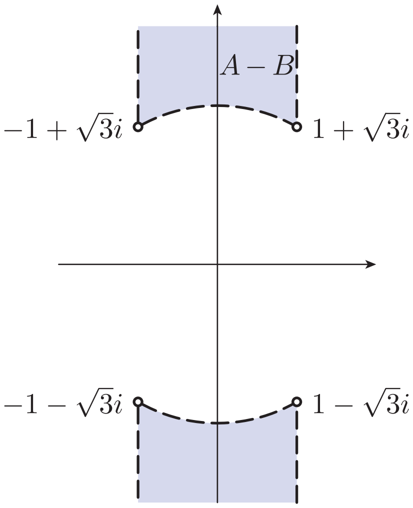
(iii) $A \cup B = \{z : |\operatorname{Re} z| < 1 \text{ or } |z| \le 2\}$



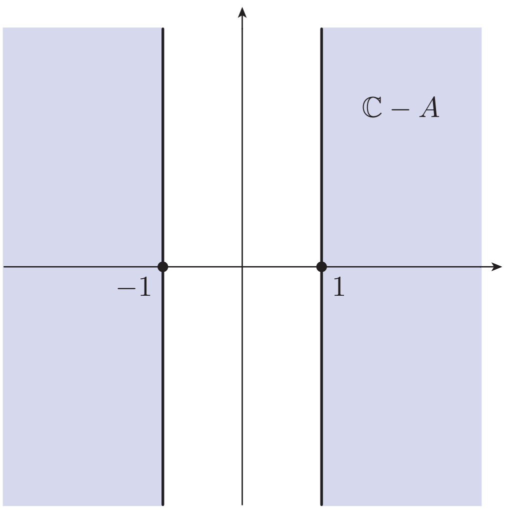
(iv) $A \cap B = \{z : |\operatorname{Re} z| < 1, |z| \leq 2\}$



(v) $A - B = \{z : |\operatorname{Re} z| < 1, |z| > 2\}$



(vi) $\mathbb{C} - A = \{z : |\operatorname{Re} z| \geq 1\}$

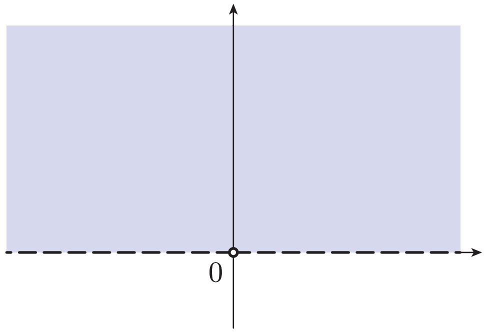


(b) $\mathbb{C} - (A \cup B)$ is the set of points z that lie in neither A nor B; that is,

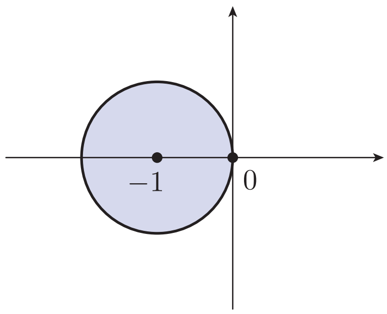
$$\mathbb{C} - (A \cup B) = \{z : |\operatorname{Re} z| \geq 1, |z| > 2\}.$$

Solution to Exercise 4.7

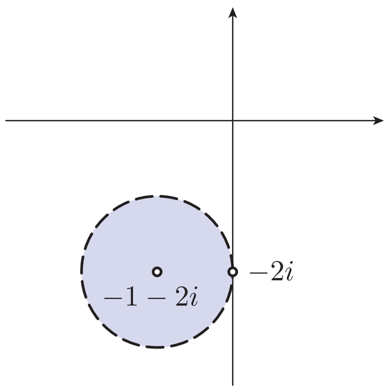
(a) $\{z : \operatorname{Im} z > 0\}$



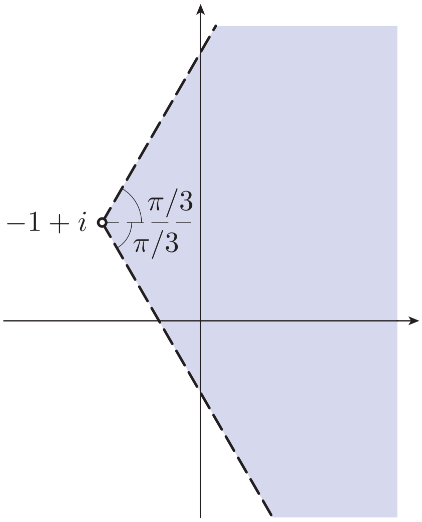
(b) $\{z : |z + 1| \leq 1\}$



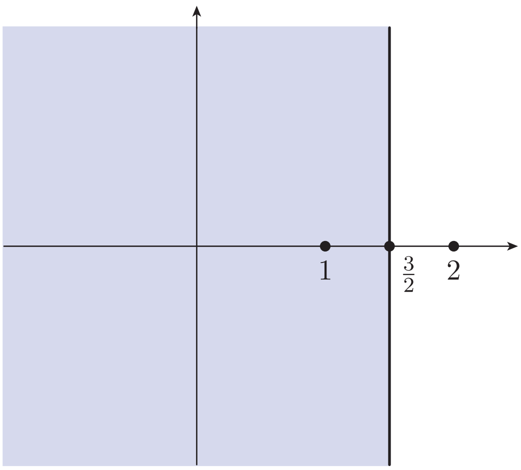
(c) $\{z : 0 < |z + 1 + 2i| < 1\}$



(d) $\{z : |\mathrm{Arg}(z + 1 - i)| < \pi/3\}$

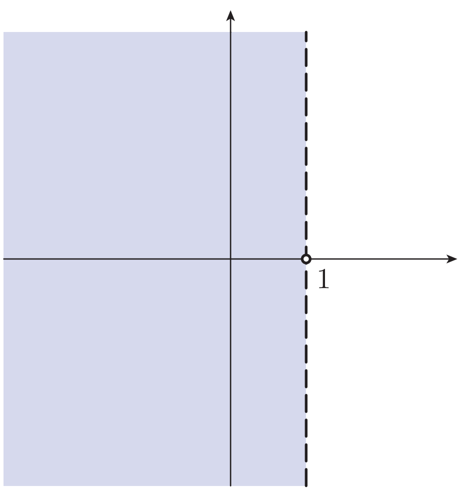


(e) $\{z : |z - 1| \leq |z - 2|\}$

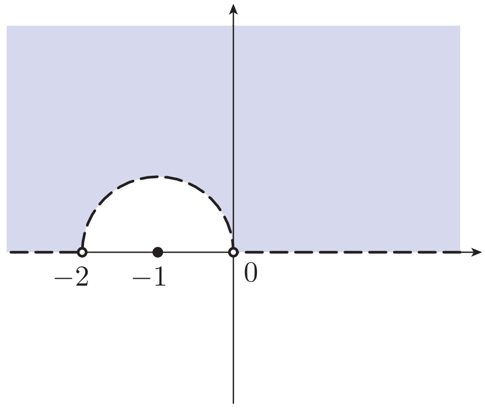


This set consists of all points z whose distance from 1 is less than or equal to its distance from 2.

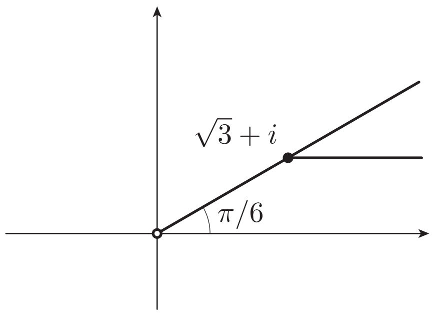
(f) $\mathbb{C} - \{z : \mathrm{Re}\, z \geq 1\}$



(g) $\{z : \mathrm{Im}\, z > 0\} - \{z : |z + 1| \leq 1\}$



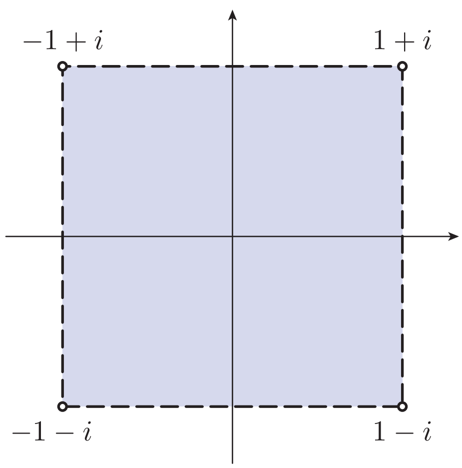
(h) $\{z : \mathrm{Arg}\, z = \pi/6\} \cup \{z : \mathrm{Arg}(z - \sqrt{3} - i) = 0\}$



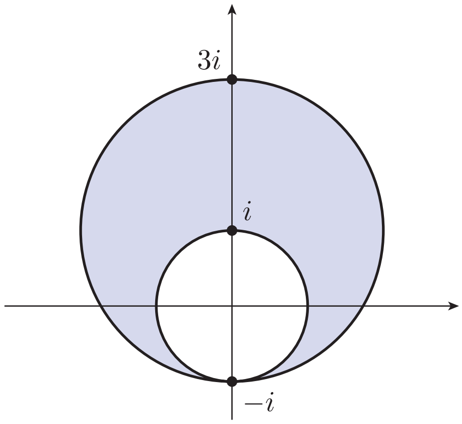
(i) The sets $\{z : \mathrm{Arg}\, z = \pi/6\}$ and $\{z : \mathrm{Arg}(z - \sqrt{3} - i) = 0\}$ have no points in common: their intersection is the empty set, $\varnothing$. We make no attempt to sketch $\varnothing$!

Solution to Exercise 4.8

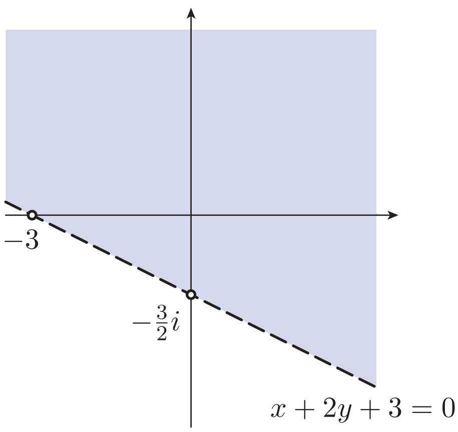
(a) $\{z : |\mathrm{Re}\, z| < 1, |\mathrm{Im}\, z| < 1\}$



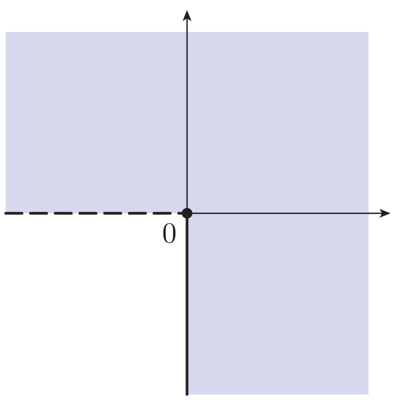
(b) $\{z : |z - i| \leq 2, |z| \geq 1\}$



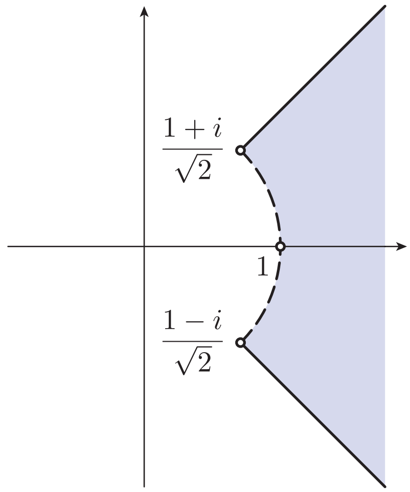
(c) $\{z : \operatorname{Re} z + 2 \operatorname{Im} z + 3 > 0\}$



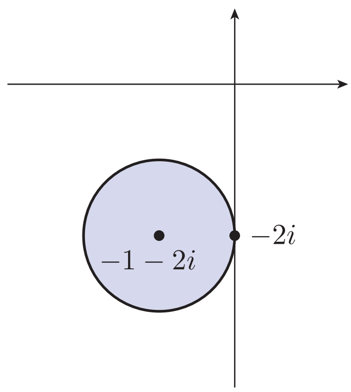
(d) $\{z : \operatorname{Re} z \geq 0\} \cup \{z : \operatorname{Im} z > 0\}$



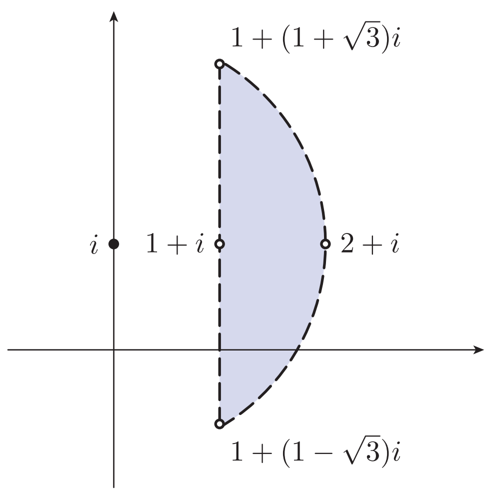
(e) $\{z : |z| > 1, |\operatorname{Arg} z| \leq \pi/4\}$



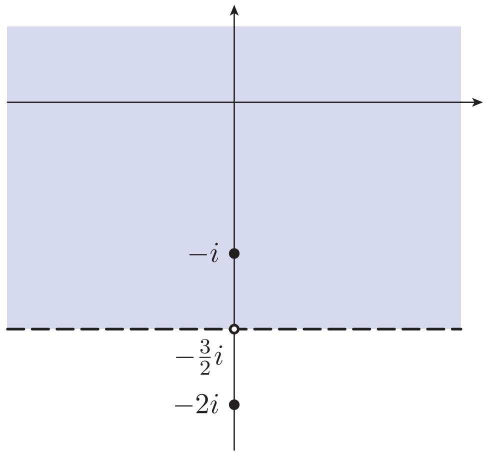
(f) $\{z : |z + 1 + 2i| \leq 1\}$



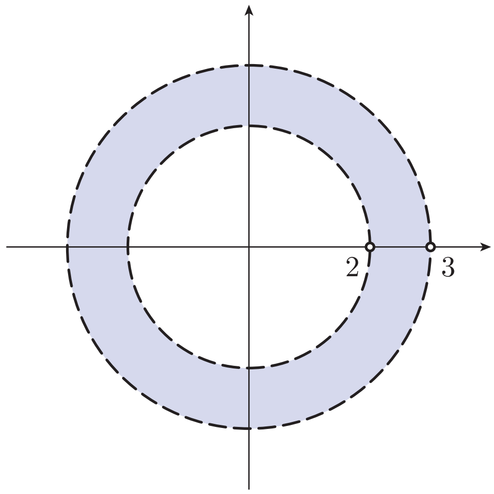
(g) $\{z : \operatorname{Re} z > 1, |z - i| < 2\}$



(h) $\{z : |z+i| < |z+2i|\}$



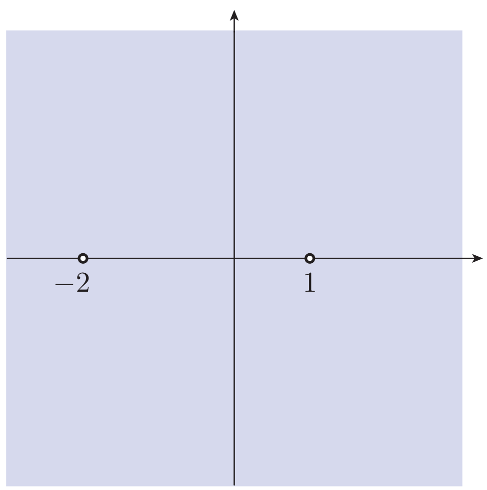
(i) $\{z : |z| < 3\} - \{z : |z| \leq 2\}$



(j) Since $z^2 + z - 2 = (z+2)(z-1)$, we see that

$$\{z : z^2 + z - 2 = 0\} = \{1, -2\},$$

so the set is $\mathbb{C} - \{1, -2\}$.



Solution to Exercise 5.1

Rearranging the given inequality, we obtain

$$\begin{aligned} \frac{3r}{r^2+2} < 1 &\iff 3r < r^2 + 2 \\ &\qquad (\text{since } r^2 + 2 > 0, \text{ for all } r) \\ &\iff 0 < r^2 - 3r + 2 \\ &\iff 0 < (r-1)(r-2). \end{aligned}$$

Since the final inequality is true for $r > 2$, the first inequality must be true for $r > 2$.

Solution to Exercise 5.2

(a) By the Triangle Inequality,

$$|3 + 4z^2| \leq |3| + |4z^2| = 3 + 4|z|^2.$$

Hence, for $|z| = 1$,

$$|3 + 4z^2| \leq 7,$$

so

$$\left| \frac{1}{3+4z^2} \right| \geq \frac{1}{7}.$$

Now, by the backwards form of the Triangle Inequality,

$$|3 + 4z^2| \geq \big| 4|z|^2 - |3| \big|.$$

Hence, for $|z| = 1$,

$$|3 + 4z^2| \geq |4 - 3| = 1,$$

so

$$\left| \frac{1}{3+4z^2} \right| \leq \frac{1}{1} = 1.$$

Thus

$$\frac{1}{7} \leq \left| \frac{1}{3+4z^2} \right| \leq 1, \quad \text{for } |z| = 1,$$

as required.

(b) We first establish the right-hand inequality. By the Triangle Inequality,

$$|z^3 + 2z + 1| \leq |z|^3 + 2|z| + 1,$$

and, by the backwards form of the Triangle Inequality,

$$|z^2 + 1| \geq \big| |z|^2 - 1 \big|.$$

Hence, for $|z| = 3$,

$$|z^3 + 2z + 1| \leq 34 \quad \text{and} \quad |z^2 + 1| \geq 8,$$

so

$$\left|\frac{z^3 + 2z + 1}{z^2 + 1}\right| \leq \frac{34}{8} = \frac{17}{4},$$

as required.

Next we establish the left-hand inequality. By the backwards form of the Triangle Inequality,

$$|z^3 + 2z + 1| \geq |z|^3 - 2|z| - 1,$$

and, by the usual form of the Triangle Inequality,

$$|z^2 + 1| \leq |z|^2 + 1.$$

Hence, for $|z| = 3$,

$$|z^3 + 2z + 1| \geq 20 \quad \text{and} \quad |z^2 + 1| \leq 10,$$

so

$$\left|\frac{z^3 + 2z + 1}{z^2 + 1}\right| \geq \frac{20}{10} = 2,$$

as required.

Solution to Exercise 5.3

Throughout this solution and the next two solutions, the appropriate version of the Triangle Inequality is given in parentheses.

(a) $|z + 3| \leq |z| + |3|$ (usual form)
$= 2 + 3 = 5$, for $|z| = 2$.

(b) $|z - 4i| \leq |z| + |4i|$ (usual form)
$= 2 + 4 = 6$, for $|z| = 2$.

(c) $|3z + 2| \leq |3z| + |2|$ (usual form)
$= 3|z| + 2$
$= 3 \times 2 + 2 = 8$, for $|z| = 2$.

(d) $|3z^2 - 5| \leq |3z^2| + |5|$ (usual form)
$= 3|z|^2 + 5$
$= 3 \times 2^2 + 5 = 17$, for $|z| = 2$.

(e) $|z^2 + z + 1| \leq |z^2| + |z| + |1|$ (usual form)
$= |z|^2 + |z| + 1$
$= 2^2 + 2 + 1 = 7$, for $|z| = 2$.

Solution to Exercise 5.4

(a) $|z - 2| \geq \big||z| - |2|\big|$ (backwards form)
$= |5 - 2| = 3$, for $|z| = 5$.

(b) $|z + 3i| \geq \big||z| - |3i|\big|$ (backwards form)
$= |5 - 3| = 2$, for $|z| = 5$.

(c) $|z - 7| \geq \big||z| - |7|\big|$ (backwards form)
$= |5 - 7| = 2$, for $|z| = 5$.

(d) $|2z - 7| \geq \big||2z| - |7|\big|$ (backwards form)
$= |10 - 7| = 3$, for $|z| = 5$.

Solution to Exercise 5.5

First we obtain upper and lower estimates for $|z^3 + 1|$, for $|z| = 4$, using appropriate versions of the Triangle Inequality. We have

$|z^3 + 1| \leq |z^3| + |1|$ (usual form)
$= |z|^3 + 1$
$= 64 + 1 = 65$, for $|z| = 4$.

Also,

$|z^3 + 1| \geq \big||z^3| - |1|\big|$ (backwards form)
$= \big||z|^3 - 1\big|$
$= |64 - 1| = 63$, for $|z| = 4$.

Next we obtain upper and lower estimates for $|z^3 - 1|$, for $|z| = 4$. We have

$|z^3 - 1| \leq |z^3| + |1|$ (usual form)
$= |z|^3 + 1$
$= 64 + 1 = 65$, for $|z| = 4$.

Also,

$|z^3 - 1| \geq \big||z^3| - |1|\big|$ (backwards form)
$= \big||z|^3 - 1\big|$
$= |64 - 1| = 63$, for $|z| = 4$.

So

$$\left|\frac{z^3 + 1}{z^3 - 1}\right| = |z^3 + 1| \times \frac{1}{|z^3 - 1|}$$
$$\leq 65 \times \frac{1}{63} = \frac{65}{63}, \quad \text{for } |z| = 4,$$

and

$$\left|\frac{z^3 + 1}{z^3 - 1}\right| = |z^3 + 1| \times \frac{1}{|z^3 - 1|}$$
$$\geq 63 \times \frac{1}{65} = \frac{63}{65}, \quad \text{for } |z| = 4.$$

Hence the given inequalities hold with $m = 63/65$ and $M = 65/63$.

Unit A2
Complex functions

Introduction

In Unit A1 we introduced complex numbers and described a number of fundamental operations using them; in particular, we investigated the solution of equations involving complex numbers. We now study *complex functions* and we find that many of the standard real functions (polynomial, rational, exponential, trigonometric and hyperbolic) have complex analogues with remarkable geometric properties.

In Section 1 we establish the basic language associated with complex functions, such as sums, products, quotients and composites of functions. We also discuss the problems associated with forming inverses of complex functions.

In Section 2 we discuss two special types of complex function, namely those with codomain $\mathbb{R}$ and those with domain a subset of $\mathbb{R}$. Each of these types of function has a role to play in our understanding of the geometric effect of a complex function, which we study in Section 3. There we examine some particular complex functions in detail and sketch the images of various 'grids' under these functions.

In Section 4 we introduce the complex exponential function and describe its geometric properties. The complex trigonometric and hyperbolic functions are then defined in terms of this exponential function.

Finally, in Section 5, we introduce the complex logarithm function, which will have an important part to play in complex integration; we also discuss complex powers.

Unit guide

You should find that Section 1 is mainly revision and you should aim to work through it fairly quickly. Sections 2 and 3 are closely related, so you might try to study them in one session. Sections 4 and 5 are also closely related, and they contain a good deal of basic technical material which will be used throughout the module.

1 Complex functions and their properties

After working through this section, you should be able to:

- determine the domain and rule of the *sum*, *product* and *quotient* of two complex functions
- determine the domain and rule of the *composite* of two complex functions
- determine whether a given complex function has an *inverse function*, and find that inverse function in suitable cases.

1.1 Defining complex functions

The main aim of complex analysis is to extend the theory of calculus to include functions of a complex variable; such functions are called *complex functions*.

Definitions

A **complex function** f is defined by specifying:

- two sets A and B in the complex plane $\mathbb{C}$
- a rule that associates with each number z in A a unique number w in B; we write $w = f(z)$.

The set A is called the **domain** of the function f, and the set B is called the **codomain** of f.

The number w is called the **image of z under f**, or the **value of f at z**, and we say that **f maps z to w**.

For example, consider the expression

$$\begin{aligned} f\colon \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto z^2, \end{aligned} \tag{1.1}$$

which defines a complex function with $A = \mathbb{C}$ and $B = \mathbb{C}$, and with rule given by $f(z) = z^2$. Under this function f, the image of $z = 2i$ is $w = f(2i) = (2i)^2 = -4$, and, similarly, the image of $1 - i$ is $(1 - i)^2 = -2i$. These values are plotted in Figure 1.1; the points $2i$ and $1 - i$ are shown in the domain (the z-plane) and the corresponding images are shown in the codomain (the w-plane).

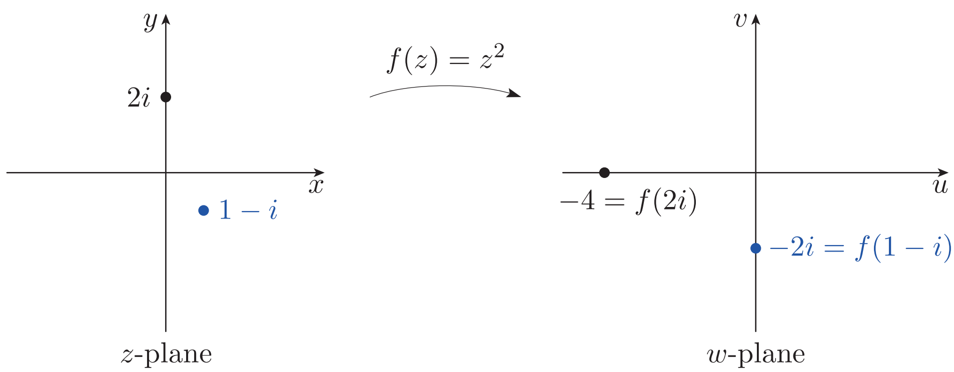


Figure 1.1 Images of points under the function $f(z) = z^2$

We will often use this type of diagrammatic representation for specific complex functions, whereas a general complex function $f\colon A \longrightarrow B$ may be represented by a diagram in which the sets A and B appear as 'blobs' (Figure 1.2).

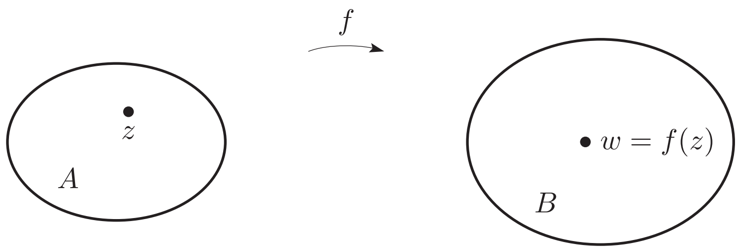


Figure 1.2 Representation of a complex function $f\colon A \longrightarrow B$

These diagrams make it clear why some texts refer to the codomain B as the *range* or *target* of f.

Observe that in Figure 1.1 we have labelled the axes of the z-plane with x and y and labelled the axes of the w-plane with u and v, to help distinguish the two planes. We retain this convention for labelling the axes of the complex plane for many of the figures in this unit (where it seems helpful). In other units, however, we do not generally label the axes of the complex plane.

Usually we will refer to a complex function f simply as a 'function', unless there is some particular reason to emphasise that f is a complex function. Other commonly used words for function are *map*, *mapping* and *transformation*.

The notation (1.1) can be written more concisely in one of the forms

$$f(z) = z^2 \quad (z \in \mathbb{C}) \quad \text{or} \quad z \longmapsto z^2 \quad (z \in \mathbb{C}),$$

where it is assumed that the codomain is $\mathbb{C}$, or in one of the forms

$$f(z) = z^2 \quad \text{or} \quad z \longmapsto z^2,$$

where it is assumed that both the domain and codomain are $\mathbb{C}$. To avoid uncertainty, we adopt the following convention.

Convention

When a function f is specified *just* by its rule, it is to be understood that the domain of f is the set of all complex numbers to which the rule is applicable, and the codomain of f is $\mathbb{C}$.

For example, the function

$$f(z) = \frac{1}{z - i}$$

has domain $\mathbb{C} - \{i\}$ because $1/(z - i)$ is defined for all complex numbers z other than $z = i$. Its codomain is $\mathbb{C}$.

Exercise 1.1

Using the convention above, write down the domain and codomain of each of the following functions.

(a) $f(z) = z + 2$ (b) $f(z) = \dfrac{z}{z+2}$ (c) $f(z) = \operatorname{Arg} z$

(d) $f(z) = \dfrac{1}{z^2+1}$

1.2 The image set of a function

If a function f has domain A and codomain B, then for each z in A the image $w = f(z)$ is in B. However, it is not necessarily true that for each w in B, there is some z in A such that $f(z) = w$. For example, if $f(z) = 1/(z - i)$, then the domain of f is $\mathbb{C} - \{i\}$ and the codomain is $\mathbb{C}$ (by the convention). However, there is no point z in the domain of f which maps to the point 0 (in the codomain); in other words, there is no z in $\mathbb{C} - \{i\}$ such that $f(z) = 0$.

Definitions

Given a function $f\colon A \longrightarrow B$, the **image set** of f, written $f(A)$, is the set of all values $f(z)$, where $z \in A$. Thus

$$f(A) = \{f(z) : z \in A\}.$$

If $f(A) = B$, then the function f is said to be **onto**.

Note that some texts use *surjective* rather than *onto*.

Figure 1.3 depicts a function that is *not* onto because $f(A) \neq B$.

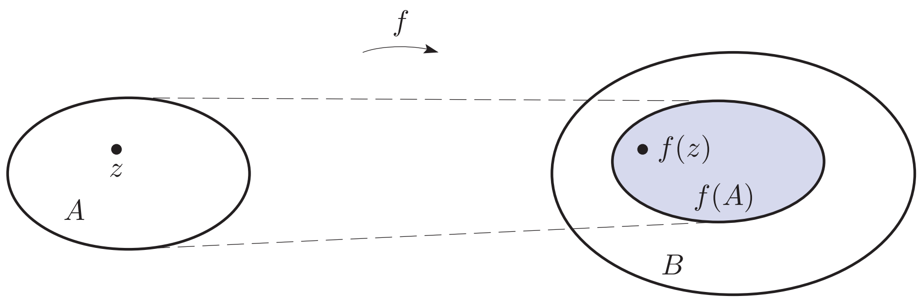


Figure 1.3 The image set of a function f

Example 1.1

Determine the image set of the function

$$f(z) = \frac{1}{z-i}.$$

Solution

The domain of f is $A = \mathbb{C} - \{i\}$ and $f(z) = \dfrac{1}{z-i}$, so we have

$$\begin{aligned} f(A) &= \left\{\frac{1}{z-i} : z \in \mathbb{C} - \{i\}\right\} \\ &= \left\{w = \frac{1}{z-i} : z \neq i\right\} \\ &= \left\{w : z = i + \frac{1}{w} \neq i\right\}, \end{aligned}$$

where the last line was obtained by rearranging $w = 1/(z-i)$ to give $z = i + 1/w$. Since $z = i + 1/w$ exists and is not equal to i if and only if $w \neq 0$, we see that

$$f(A) = \{w : w \neq 0\} = \mathbb{C} - \{0\}.$$

In the preceding example, we found $f(A)$ by defining $w = f(z)$ and then expressing $f(A)$ in the form $\{w : \text{condition on } w\}$. This is a useful strategy to adopt when approaching similar examples (even if you can 'see' what the image set is, as you may have done in Example 1.1).

Exercise 1.2

For each of the following functions f, determine the image set of f.

(a) $f(z) = 3iz$ (b) $f(z) = \dfrac{3z+1}{z+i}$ (c) $f(z) = \operatorname{Im} z$

Several important functions, such as

$$z \longmapsto \operatorname{Re} z, \quad z \longmapsto \operatorname{Im} z, \quad z \longmapsto |z|, \quad z \longmapsto \operatorname{Arg} z,$$

have image sets that are subsets of the real line. For example, the function $f(z) = \operatorname{Re} z$ has image set $f(\mathbb{C}) = \mathbb{R}$, the whole real line (Figure 1.4). Such functions are called *real-valued functions*. Be careful not to confuse real-valued functions with *real functions*, which are functions whose domain and codomain consist of real numbers.

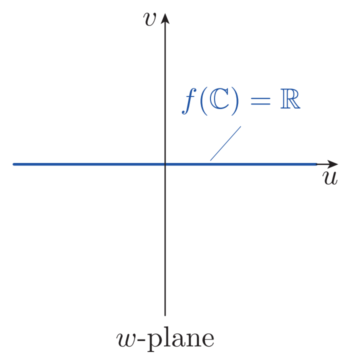


Figure 1.4 The image set of $f(z) = \operatorname{Re} z$

Definitions

A function $f\colon A \longrightarrow B$ is called a **real-valued function** (of a complex variable) if $f(A) \subseteq \mathbb{R}$.

The function f is called a **real function** if $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$.

Exercise 1.3

Write down the image sets of each of the following functions.

(a) $f(z) = |z|$ (b) $f(z) = \operatorname{Arg} z$

1.3 Sums, products and quotients of functions

Let f and g be the functions

$$f(z) = \frac{1}{z} \quad (z \in \mathbb{C} - \{0\}) \quad \text{and} \quad g(z) = z^2 + 1 \quad (z \in \mathbb{C}).$$

The expressions $f + g$, fg and f/g are used to denote the following functions:

$$\begin{aligned}
(f+g)(z) &= f(z) + g(z) = \frac{1}{z} + (z^2+1) && (z \in \mathbb{C} - \{0\}),\\
(fg)(z) &= f(z)g(z) = \frac{1}{z}(z^2+1) && (z \in \mathbb{C} - \{0\}),\\
(f/g)(z) &= f(z)/g(z) = \frac{1}{z}\left(\frac{1}{z^2+1}\right) && (z \in \mathbb{C} - \{0, i, -i\}).
\end{aligned}$$

The domains of $f + g$ and fg include only those points at which both f and g are defined. When forming the quotient f/g, we must also exclude from the domain those points z at which $g(z) = 0$. The points at which a function takes the value zero are called the **zeros** of the function.

Definitions

Let $f\colon A \longrightarrow \mathbb{C}$ and $g\colon B \longrightarrow \mathbb{C}$ be functions.

The **sum** $f + g$ is the function with domain $A \cap B$ and rule

$$(f+g)(z) = f(z) + g(z).$$

The **multiple** λf, where $\lambda \in \mathbb{C}$, is the function with domain A and rule

$$(\lambda f)(z) = \lambda f(z).$$

The **product** fg is the function with domain $A \cap B$ and rule

$$(fg)(z) = f(z)g(z).$$

The **quotient** f/g is the function with domain $A \cap B - \{z : g(z) = 0\}$ and rule

$$(f/g)(z) = f(z)/g(z).$$

The next exercise gives you practice at combining functions.

Exercise 1.4

Let f and g be the functions

$$f(z) = \frac{1}{z} \quad (z \in \mathbb{C} - \{0\}) \quad \text{and} \quad g(z) = \frac{z+3i}{z^2 - z} \quad (z \in \mathbb{C} - \{0, 1\}).$$

Determine the domain and rule of each of the following functions.

(a) $f + g$ (b) fg (c) f/g

Starting from the two basic functions $z \longmapsto 1$ and $z \longmapsto z$, we can build up any **polynomial function** of degree n

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

where $a_0, a_1, \ldots, a_n \in \mathbb{C}, a_n \neq 0$, by forming suitable sums, multiples and products. The domain of any polynomial function is $\mathbb{C}$. Allowing quotients also, we can build up any **rational function**, that is, any function of the form

$$f(z) = \frac{p(z)}{q(z)},$$

where p and q are polynomial functions. It follows from the definition of quotient that the domain of such a rational function is

$$\mathbb{C} - \{z : q(z) = 0\}.$$

For example, the rational function

$$f(z) = \frac{z}{z^2 + 1}$$

has domain $\mathbb{C} - \{i, -i\}$.

1.4 Composite functions

Let f and g be functions. The composite function $g \circ f$ is obtained by applying first f and then g. Thus the rule of $g \circ f$ is

$$(g \circ f)(z) = g(f(z)).$$

The process of forming $g \circ f$ is called ‘composition of functions’ or ‘composing functions’.

For example, if

$$f(z) = \frac{1}{z} \quad (z \in \mathbb{C} - \{0\}) \quad \text{and} \quad g(z) = z^2 + 1 \quad (z \in \mathbb{C}),$$

then the rule of $g \circ f$ is

$$(g \circ f)(z) = g(1/z) = (1/z)^2 + 1,$$

whereas the rule of $f \circ g$ is

$$(f \circ g)(z) = f(z^2 + 1) = 1/(z^2 + 1).$$

But what is the domain of a composite function? In general, if $f\colon A \longrightarrow \mathbb{C}$ and $g\colon B \longrightarrow \mathbb{C}$, then the value $g(f(z))$ is defined if and only if

z lies in A and $f(z)$ lies in B.

Thus if z_1 and z_2 are elements of A such that $f(z_1) \in B$ but $f(z_2) \notin B$, then $g(f(z))$ is defined for $z = z_1$ but not for $z = z_2$, as indicated in Figure 1.5.

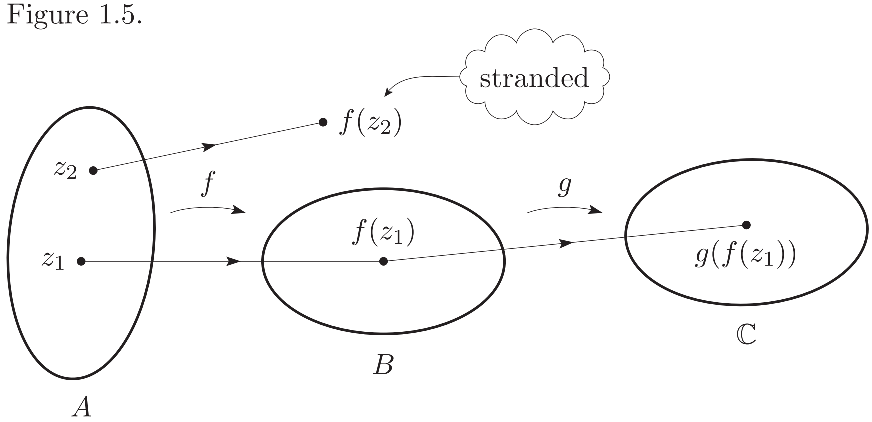


Figure 1.5 Composition of two functions f and g

We define the domain of $g \circ f$ to be consistent with this.

Definition

Let $f\colon A \longrightarrow \mathbb{C}$ and $g\colon B \longrightarrow \mathbb{C}$ be complex functions. Then the **composite function** $g \circ f$ has domain

$$\{z \in A : f(z) \in B\}$$

and rule

$$(g \circ f)(z) = g(f(z)).$$

To find the domain of $g \circ f$, we remove from A (the domain of f) each point whose image under f is not in B (the domain of g). Thus the domain of $g \circ f$ can be written as

$$A - \{z : f(z) \notin B\}.$$

For example, if

$$f(z) = z^2 + i \quad (z \in \mathbb{C}) \quad \text{and} \quad g(z) = \frac{1}{z - i} \quad (z \in \mathbb{C} - \{i\}),$$

then the domain of $g \circ f$ is

$$\mathbb{C} - \{z : z^2 + i \notin \mathbb{C} - \{i\}\} = \mathbb{C} - \{z : z^2 + i = i\} = \mathbb{C} - \{0\}.$$

In practice, it often happens that the image set of f is contained in B (that is, $f(A) \subseteq B$) and in this case the domain of $g \circ f$ is A itself. (This always happens when the domain of g is $\mathbb{C}$, of course.) For example, if

$$f(z) = \frac{1}{z - i} \quad (z \in \mathbb{C} - \{i\}) \quad \text{and} \quad g(z) = \operatorname{Arg} z \quad (z \in \mathbb{C} - \{0\}),$$

then $A = \mathbb{C} - \{i\}$, $B = \mathbb{C} - \{0\}$. Also, as you saw in Example 1.1,

$$f(\mathbb{C} - \{i\}) = \mathbb{C} - \{0\},$$

which is (contained in) the domain of g. Thus the domain of $g \circ f$ is $\mathbb{C} - \{i\}$. In fact, $g(f(z)) = \operatorname{Arg}(1/(z-i))$ and the only complex number z for which $\operatorname{Arg}(1/(z-i))$ is not defined is $z = i$.

We remark that, in contrast to our approach, some texts *require* $f(A) \subseteq B$ in the definition of $g \circ f$.

Exercise 1.5

Let f and g be the functions

$$f(z) = \frac{1}{z} \quad (z \in \mathbb{C} - \{0\}) \quad \text{and} \quad g(z) = \frac{z+3i}{z^2 - z} \quad (z \in \mathbb{C} - \{0,1\}).$$

Determine the domain and rule of each of the following functions.

(a) $g \circ f$ (b) $f \circ g$

1.5 Inverse functions

Let f be the function

$$f(z) = 3z \quad (z \in \mathbb{C}).$$

Then for each number w in $\mathbb{C}$, there is a unique number $z = \frac{1}{3}w$ in the domain of f such that

$$f(z) = f\left(\tfrac{1}{3}w\right) = 3 \times \tfrac{1}{3}w = w.$$

The corresponding function $g\colon w \longmapsto \frac{1}{3}w$ is called the *inverse function* of f because it 'undoes' the effect of f. To be precise,

$$g(f(z)) = \tfrac{1}{3}f(z) = \tfrac{1}{3} \times 3z = z \quad (z \in \mathbb{C}).$$

Similarly, f undoes g:

$$f(g(w)) = 3g(w) = 3 \times \tfrac{1}{3}w = w \quad (w \in \mathbb{C}).$$

The inverse function of f is denoted by f^{-1} (see Figure 1.6).

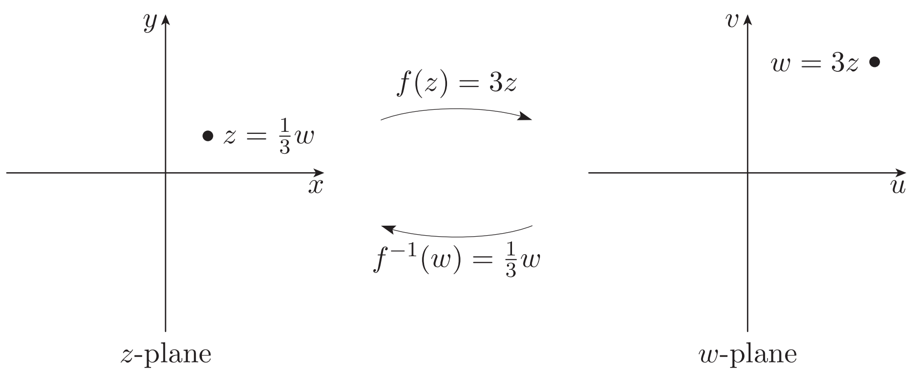


Figure 1.6 Images of points under the functions $f(z) = 3z$ and $f^{-1}(w) = \frac{1}{3}w$

Not every function has an inverse function. For example, consider the function

$$f(z) = z^2 \quad (z \in \mathbb{C}).$$

Since $f(2) = 4$ and $f(-2) = 4$, we cannot assign a unique value z in the domain of f such that $f(z) = 4$ (Figure 1.7).

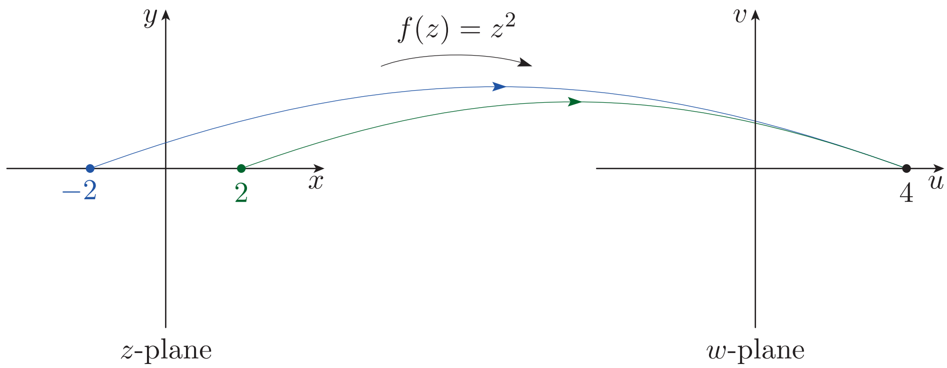


Figure 1.7 Images of the points 2 and -2 under the function $f(z) = z^2$

The problem here is that the function f is not *one-to-one*. In general, it is possible to define the inverse of a function only if that function is one-to-one.

Definition

The function $f\colon A \longrightarrow B$ is **one-to-one** if the images under f of distinct points in A are also distinct; that is,

$$\text{if } z_1, z_2 \in A \text{ and } z_1 \neq z_2, \text{ then } f(z_1) \neq f(z_2).$$

Note that some texts use *injective* rather than *one-to-one*.

An equivalent statement of the one-to-one condition is that

$$\text{if } w \in f(A), \text{ then there is a } \textit{unique } z \text{ in } A \text{ such that } f(z) = w.$$

This principle is illustrated in Figure 1.8, in which there is only one point z in A whose image under f is w.

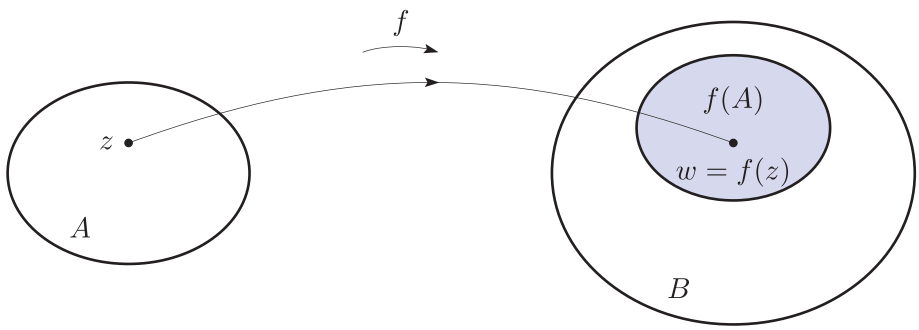


Figure 1.8 A unique point z in A such that $w = f(z)$

The uniqueness statement of the second version of the one-to-one condition makes it possible to define the inverse function f^{-1} of f with domain $f(A)$. This is illustrated in Figure 1.9.

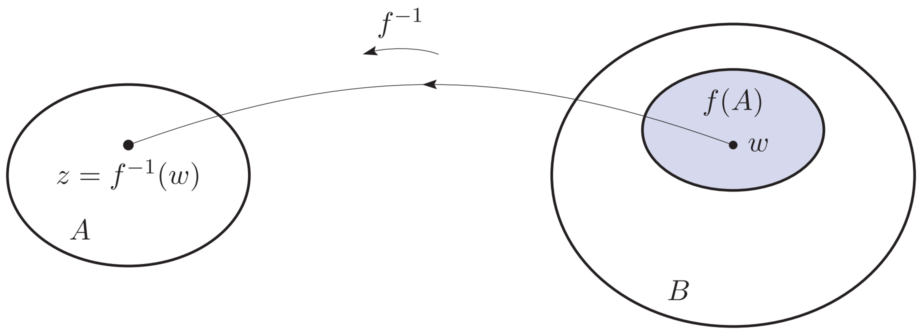


Figure 1.9 The inverse function $f^{-1}\colon f(A) \longrightarrow A$

Definition

Let $f\colon A \longrightarrow B$ be a one-to-one function. Then the **inverse function** f^{-1} of f has domain $f(A)$ and rule

$$f^{-1}(w) = z,$$

where $w = f(z)$.

Thus there are two ways of proving that a function has an inverse function.

Strategy for proving that an inverse function exists

To prove that a function f has an inverse function:

- *either* prove that f is one-to-one directly by showing that if $z_1 \neq z_2$, then $f(z_1) \neq f(z_2)$
- *or* determine the image set $f(A)$ and show that for each $w \in f(A)$ there is a unique $z \in A$ such that $f(z) = w$.

Notice that the statement of the first strategy 'if $z_1 \neq z_2$, then $f(z_1) \neq f(z_2)$' is equivalent to

$$f(z_1) = f(z_2) \implies z_1 = z_2.$$

For the second strategy, one way of demonstrating that 'there is a unique $z \in A$ such that $f(z) = w$' is to *find* the rule for f^{-1} (if this is possible). For some functions this can be done by solving the equation $w = f(z)$ to obtain a unique z in terms of w. We adopt the second strategy (rather than the first strategy) with this approach whenever it is possible, because it has the advantage that we thereby specify the function f^{-1}. This strategy is illustrated in the next example.

Example 1.2

Prove that the function

$$f(z) = \frac{1}{z-i} \quad (z \in \mathbb{C} - \{i\})$$

has an inverse function, and determine the domain and rule of f^{-1}.

Solution

First we determine the image set of f. This is

$$f(\mathbb{C} - \{i\}) = \mathbb{C} - \{0\} \quad \text{(from Example 1.1).}$$

Now, for each $w \in \mathbb{C} - \{0\}$, we wish to solve the equation

$$w = \frac{1}{z-i}$$

to obtain a *unique* solution z in $\mathbb{C} - \{i\}$. This is achieved by the rearrangement

$$z = i + \frac{1}{w}.$$

Thus f is a one-to-one function, with image set $\mathbb{C} - \{0\}$. Hence f has an inverse function f^{-1} with domain $\mathbb{C} - \{0\}$ and rule

$$f^{-1}(w) = i + \frac{1}{w}.$$

Remark

Usually when defining a function, we write z for the domain variable. To conform to this practice, we could rewrite the inverse function f^{-1} in Example 1.2 in the form

$$f^{-1}(z) = i + \frac{1}{z} \quad (z \in \mathbb{C} - \{0\}),$$

with z in place of w, *after* the required algebraic manipulations have been completed. It is *not* necessary to do this as a matter of routine.

Exercise 1.6

Prove that the function

$$f(z) = \frac{3z+1}{z+i} \quad (z \in \mathbb{C} - \{-i\})$$

has an inverse function f^{-1}, and determine the domain and rule of f^{-1}. (See Exercise 1.2(b).)

As we pointed out earlier, the function $f(z) = z^2$ is not one-to-one (on $\mathbb{C}$), so it does not have an inverse function. One way round this difficulty is to

reduce the domain of f (without changing the rule) so as to make the resulting function one-to-one. Note that reducing the domain of a function leads to a *new* function – a **restriction** of the original function – which should really be denoted by a different letter. In practice, we usually retain the same letter, particularly if the original domain is no longer under discussion. Here is an example of this.

Example 1.3

Let $A = \{0\} \cup \{z : -\pi/2 < \operatorname{Arg} z \le \pi/2\}$, as shown in Figure 1.10. Prove that the function

$$f(z) = z^2 \quad (z \in A)$$

has an inverse function f^{-1}, and determine the domain and rule of f^{-1}.

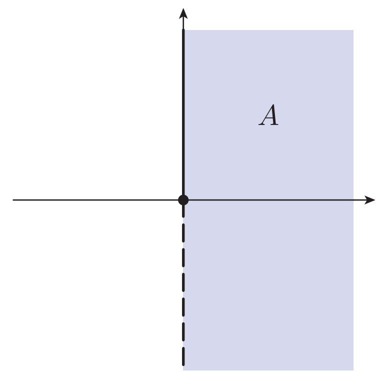


Figure 1.10 The set $A = \{0\} \cup \{z : -\pi/2 < \operatorname{Arg} z \le \pi/2\}$

Solution

Let us first determine the image set of f. By writing $z = r(\cos\theta + i\sin\theta)$, we obtain

$$\begin{aligned} f(A) &= \{z^2 : z \in A\} \\ &= \{0\} \cup \left\{w = z^2 : -\pi/2 < \operatorname{Arg} z \le \pi/2\right\} \\ &= \{0\} \cup \left\{w = r^2(\cos 2\theta + i\sin 2\theta) : r > 0,\ -\pi/2 < \theta \le \pi/2\right\} \\ &= \{0\} \cup \{w = \rho(\cos\phi + i\sin\phi) : \rho > 0,\ -\pi < \phi \le \pi\}, \end{aligned}$$

where $\rho = r^2$ and $\phi = 2\theta$. Thus $f(A) = \mathbb{C}$.

Now, for each $w \in \mathbb{C}$, we wish to solve the equation

$$w = z^2 \tag{1.2}$$

to obtain a *unique* solution z in A. If $w = 0$, then equation (1.2) has the unique solution $z = 0$. If $w \ne 0$, then w can be written in the form

$$w = \rho(\cos\phi + i\sin\phi),$$

where $\rho > 0$ and $-\pi < \phi \le \pi$, and equation (1.2) has exactly two solutions:

$$\begin{aligned} z_0 &= \rho^{1/2}(\cos(\phi/2) + i\sin(\phi/2)), \\ z_1 &= \rho^{1/2}(\cos(\phi/2 + \pi) + i\sin(\phi/2 + \pi)) \end{aligned}$$

(by Theorem 3.1 of Unit A1), which are shown in Figure 1.11. Clearly, $z_0 \in A$, since $-\pi/2 < \phi/2 \le \pi/2$, whereas $z_1 \notin A$.

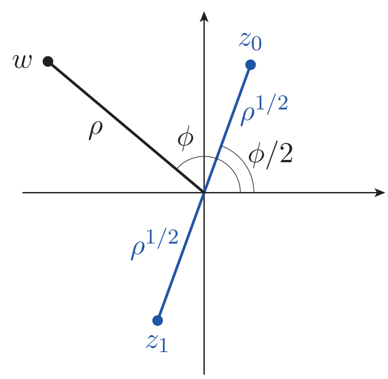


Figure 1.11 The two solutions of $w = z^2$

Thus f is a one-to-one function, with image set $\mathbb{C}$. Hence f has an inverse function f^{-1} with domain $\mathbb{C}$ and rule given by $f^{-1}(0) = 0$ and, for $w \ne 0$,

$$f^{-1}(w) = \rho^{1/2}(\cos(\phi/2) + i\sin(\phi/2)),$$

where $w = \rho(\cos\phi + i\sin\phi)$, $\rho > 0$, $-\pi < \phi \le \pi$.

Remark

In this solution we chose ϕ to satisfy $-\pi < \phi \leq \pi$ so that $-\pi/2 < \phi/2 \leq \pi/2$, and hence $z_0 \in A$. Since $\phi = \operatorname{Arg} w$, for $w \neq 0$, it follows that z_0 is the principal square root $\sqrt{w}$ of w. Furthermore, because we defined $\sqrt{0} = 0$ (see Subsection 3.1 of Unit A1), the rule for f^{-1} can be written in the form

$$f^{-1}(w) = \sqrt{w} \quad (w \in \mathbb{C}).$$

The set A in Example 1.3 is not the only one on which the function $z \longmapsto z^2$ is one-to-one with image set $\mathbb{C}$. In the following exercise, you are asked to investigate another such set.

Exercise 1.7

Let $A = \{0\} \cup \{z : 0 \leq \operatorname{Arg} z < \pi\}$. Prove that the function

$$f(z) = z^2 \quad (z \in A)$$

has an inverse function f^{-1}, and determine the domain and rule of f^{-1}.

Further exercises

Exercise 1.8

Write down the domain of each of the following functions.

(a) $f(z) = (z-1)^2$ (b) $f(z) = \dfrac{1}{z-1}$ (c) $f(z) = \dfrac{z}{z^2+1}$

(d) $f(z) = \dfrac{1}{\operatorname{Re} z}$ (e) $f(z) = \dfrac{1}{|z|-1}$ (f) $f(z) = \dfrac{1}{z^3+1}$

Exercise 1.9

Determine the image set of each of the following functions.

(a) $f(z) = 2z+1$ (b) $f(z) = \dfrac{1}{z-1}$ (c) $f(z) = \dfrac{z}{z-1}$

(d) $f(z) = |z-1|$ (e) $f(z) = \operatorname{Re}(z+i)$ (f) $f(z) = |\operatorname{Arg} z|$

Exercise 1.10

Let

$$f(z) = \frac{z-1}{z} \quad \text{and} \quad g(z) = \frac{z}{z-1}.$$

Determine the domain and rule of each of the following functions.

(a) $f+g$ (b) $3f-2ig$ (c) fg (d) f/g

Exercise 1.11

For the functions f and g of Exercise 1.10, write down the domain and rule of each of the following functions.

(a) $f \circ g$ (b) $g \circ f$ (c) $f \circ f$

Exercise 1.12

Determine whether or not each of the functions f in Exercise 1.9 is one-to-one, and write down the inverse function of f, where possible.

Exercise 1.13

Let $A = \{0\} \cup \{z : -\pi/3 < \operatorname{Arg} z \leq \pi/3\}$. Prove that the function

$$f(z) = z^3 \quad (z \in A)$$

has an inverse function f^{-1}, and determine the domain and rule of f^{-1}.

2 Special types of complex function

After working through this section, you should be able to:

- find the *real* and *imaginary parts* of a complex function
- sketch a *path*
- obtain (where possible) the equation of a path by eliminating the parameter from its parametrisation
- find the *image* under a function of a path in simple cases
- use the table of standard parametrisations.

2.1 Real-valued functions

In Section 1 we pointed out that various common functions, such as $z \longmapsto |z|$, have image sets in $\mathbb{R}$ and are therefore called *real-valued functions*. Because the image sets of such functions lie in $\mathbb{R}$, we do not have to resort to the z-plane/w-plane representation of the functions. In fact, we can sketch the graph of a real-valued function by introducing a third axis, which is at right angles to the complex plane (z-plane). We call this third axis the *s-axis* (rather than the z-axis because of possible confusion with the complex variable $z = x + iy$). The graph of the function $z \longmapsto |z|$ is shown in Figure 2.1; it is the surface with equation $s = |z| = \sqrt{x^2 + y^2}$.

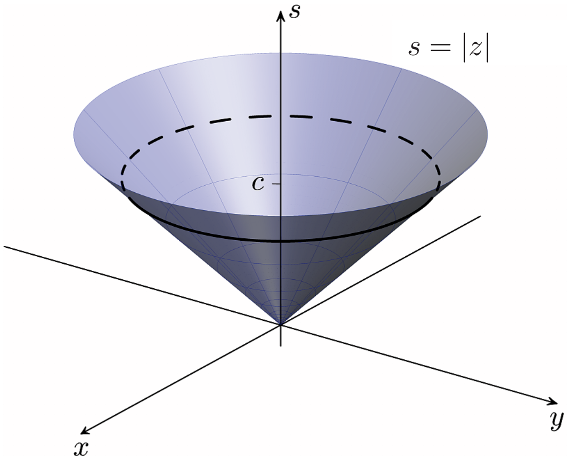


Figure 2.1 Graph of $s = |z| = \sqrt{x^2 + y^2}$

The surface is an infinite cone with apex at the origin and with axis the positive s-axis. Any horizontal plane with equation $s = c$, where c is a positive constant, intersects the cone in a circle of radius c at height c above the (x, y)-plane, as shown in Figure 2.1.

Notice that the equation $s = c$ represents a *plane* in three dimensions – the plane perpendicular to the s-axis through the point $(0, 0, c)$ – and likewise the equations $x = c$ and $y = c$ also represent planes in three dimensions. In contrast, in the two-dimensional complex plane, the equation $x = c$ represents the *line* parallel to the imaginary axis made up of complex numbers with real part c.

Figure 2.2 shows the spiral-like surface $s = \operatorname{Arg} z$. It is obtained by lifting (that is, translating vertically) each of the rays $\{z : \operatorname{Arg} z = c\}$, where $-\pi < c \leq \pi$, in the z-plane to height c. Two such lifted rays, with $c = \pi/3$ and $c = -\pi/4$, are shown in the figure. Each lifted ray is the intersection of the surface with a plane with equation $s = c$, for some constant c.

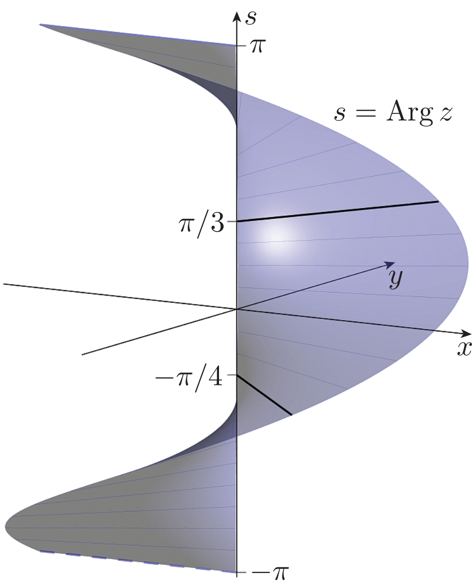


Figure 2.2 Graph of $s = \operatorname{Arg} z$

Sketching such surfaces is rather demanding from an artistic point of view, and we do not expect you to be skilled at it. Do not spend more than a few minutes on the following exercise.

Exercise 2.1

Sketch the following surfaces.

(a) $s = \operatorname{Re} z$ (b) $s = \operatorname{Im} z$

Real-valued functions of a complex variable arise naturally when we study complex functions. For example, if $f(z) = z^2$, where $z = x + iy$, then

$$f(z) = (x + iy)^2 = (x^2 - y^2) + 2ixy;$$

thus

$$f(z) = u + iv,$$

where

$$u = x^2 - y^2, \quad v = 2xy. \tag{2.1}$$

Here both $z \longmapsto u$ and $z \longmapsto v$ are real-valued functions of the complex variable z.

In general, for any complex function f, we can write $f(z)$ in the form

$$f(z) = u + iv,$$

where u and v are real. The values of these real numbers change as z changes, giving rise to two real-valued functions $z \longmapsto u$ and $z \longmapsto v$ with the same domain as f. They are called the **real** and **imaginary parts** of f, written $\operatorname{Re} f$ and $\operatorname{Im} f$, respectively. Using the notation for the real and imaginary parts of a complex number introduced in Unit A1, we have

$$\operatorname{Re} f\colon z \longmapsto \operatorname{Re}(f(z)) \quad \text{and} \quad \operatorname{Im} f\colon z \longmapsto \operatorname{Im}(f(z)).$$

Thus

$$(\operatorname{Re} f)(z) = \operatorname{Re}(f(z)) \quad \text{and} \quad (\operatorname{Im} f)(z) = \operatorname{Im}(f(z)).$$

Exercise 2.2

Determine the functions $\operatorname{Re} f$ and $\operatorname{Im} f$ for the function $f(z) = 1/z$.

If we are given two real-valued functions g and h with the same domain A in $\mathbb{C}$, then we can combine them to obtain a function $f\colon A \longrightarrow \mathbb{C}$ by writing

$$f(z) = g(z) + ih(z) \quad (z \in A).$$

For example, consider

$$g(z) = \log|z|$$

(where $\log x$ denotes the real-valued logarithm to base e of a positive number x, sometimes written as $\ln x$ or $\log_e x$ in other texts) and

$$h(z) = \operatorname{Arg} z.$$

Both g and h are real-valued functions with domain $\mathbb{C} - \{0\}$, and the function

$$f(z) = g(z) + ih(z) = \log|z| + i \operatorname{Arg} z$$

also has domain $\mathbb{C} - \{0\}$.

Figure 2.2 gave the graph of the surface $s = \operatorname{Arg} z$, and the graph of the surface $s = \log|z|$ is shown in Figure 2.3; for each real number c, the horizontal plane $s = c$ intersects the surface in a circle of radius e^c. This is because

$$s = c \text{ and } s = \log|z| \iff \log|z| = c \iff |z| = e^c.$$

We will discuss the function $f(z) = \log|z| + i \operatorname{Arg} z$ and some of its properties later in the unit.

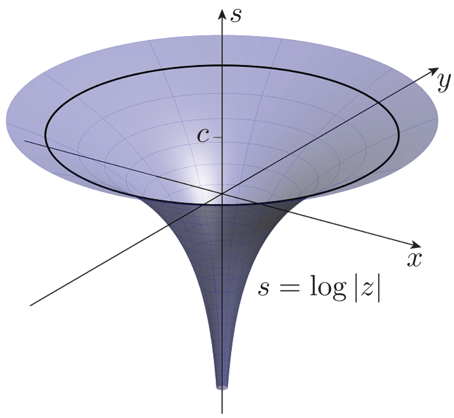


Figure 2.3 Graph of $s = \log|z|$

2.2 Functions with domains in the real numbers

In order to gain insight into the geometric effect of a given complex function f, it is helpful to be able to picture how the image point $w = f(z)$ behaves as z moves around the domain of f. This subsection is about making these intuitive geometric ideas precise.

When a point moves in a plane, it traces a *curve* or *path* as time passes. The position of the point on this path can be described by giving both the x- and y-coordinates of the point as functions of time, t. In this context, the real variable t is called a *parameter*.

For example, suppose that the x- and y-coordinates are given by the equations

$$x = \cos t, \quad y = \sin t \quad (t \in [0, 2\pi]).$$

Then, as time t increases from 0 to 2π, the point

$$z = x + iy = \cos t + i \sin t$$

moves around the circle Γ with centre 0 and radius 1, starting (when $t = 0$) and finishing (when $t = 2\pi$) at the point 1, as indicated in Figure 2.4.

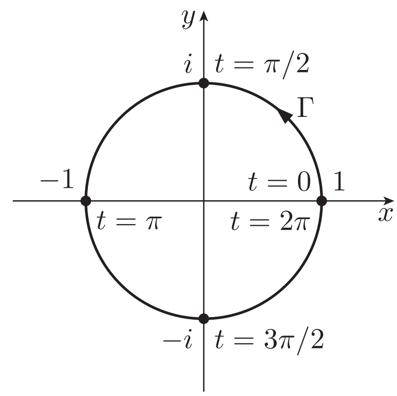


Figure 2.4 Positions of the point $z = \cos t + i \sin t$

If we introduce the function

$$\gamma(t) = \cos t + i \sin t \quad (t \in [0, 2\pi]),$$

then the circle Γ is the image set of γ; that is, $\Gamma = \gamma([0, 2\pi])$. The function γ describes a mode of traversing the circle Γ. In general, a set Γ and the associated function γ are the ingredients in our definition of the term 'path'.

Definitions

A **path** is a subset Γ of $\mathbb{C}$ which is the image set of an associated continuous function $\gamma \colon I \longrightarrow \mathbb{C}$, where I is a real interval. In this context, the function γ is called a **parametrisation** (of Γ). If

$$\gamma(t) = \phi(t) + i\psi(t) \quad (t \in I),$$

where ϕ and ψ are real functions, then the equations

$$x = \phi(t), \quad y = \psi(t) \quad (t \in I)$$

are called **parametric equations** (of Γ).

If I is the closed interval $[a, b]$, then $\gamma(a)$ and $\gamma(b)$ are called the **initial point** and **final point** of Γ, respectively.

Figure 2.5 illustrates these definitions in the case of a closed interval $I = [a, b]$.

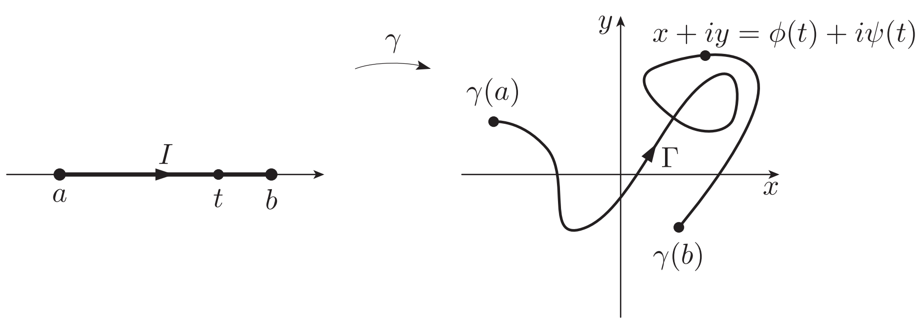


Figure 2.5 A path Γ with parametrisation $\gamma \colon I \longrightarrow \mathbb{C}$

Remarks

1. We often speak of 'the path Γ' without referring specifically to the associated parametrisation γ. Sometimes it is convenient to speak of 'the path $\Gamma : \gamma(t) = \ldots$'.
2. The condition that the function $\gamma\colon I \longrightarrow \mathbb{C}$ be continuous is included to ensure that the path Γ has no gaps in it. We will define the notion of continuity precisely in Unit A3, but meanwhile we point out that all functions γ considered in this context *are* continuous.
3. As in Figures 2.4 and 2.5, a path Γ is usually marked with an arrow (or arrows, if necessary) to show the direction in which it is traversed. (The arrow points in the direction of increasing values of t.)
4. Observe that the initial point and final point can be equal. For example, if

   $$\gamma(t) = \cos t + i \sin t \quad (t \in [0, 2\pi]),$$

   then the initial point $\gamma(0)$ and the final point $\gamma(2\pi)$ both equal 1, as shown in Figure 2.4.
5. It is sometimes possible to eliminate the parameter t from the parametric equations to obtain the equation of the path in terms of x and y alone. For example, if

   $$x = \cos t, \quad y = \sin t \quad (t \in [0, 2\pi]),$$

   then

   $$x^2 + y^2 = \cos^2 t + \sin^2 t = 1.$$

   However, unlike the parametric equations, the equation $x^2 + y^2 = 1$ does not tell us, for example, which way the arrow goes on the path.
6. It is often useful to plot a few points of the path to help us to understand the shape of a given path. This is done in the following example.

Example 2.1

Let $\gamma(t) = t^2 + it^3 \quad (t \in \mathbb{R})$.

Plot the points $\gamma(-1)$, $\gamma\left(-\frac{1}{2}\right)$, $\gamma(0)$, $\gamma\left(\frac{1}{2}\right)$, $\gamma(1)$ and hence sketch the path Γ with parametrisation γ. Determine the equation of the path Γ in terms of x and y.

Solution

First we compile a table of the required values of $x = t^2$ and $y = t^3$.

t	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
x	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1
y	-1	$-\frac{1}{8}$	0	$\frac{1}{8}$	1

We plot the points $x + iy$ and hence sketch the path Γ in Figure 2.6.

In order to eliminate t from the parametric equations $x = t^2$ and $y = t^3$, we note that

$$x^3 = (t^2)^3 = t^6 = (t^3)^2 = y^2,$$

so Γ has the equation $y^2 = x^3$.

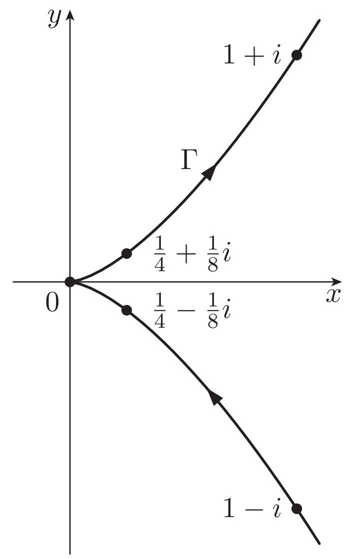


Figure 2.6 The top half of Γ has equation $y = x^{3/2}$, whereas the bottom half has equation $y = -x^{3/2}$

In the following exercise, you are asked to sketch various paths. For each path, you may find it helpful to compile a table of values (as we did in Example 2.1), but you may find that you can just use the equation of the path in terms of x and y to create your sketch.

Exercise 2.3

Sketch the paths Γ with the following parametrisations.

(a) $\gamma(t) = 1 + it \quad (t \in \mathbb{R})$

(b) $\gamma(t) = t^2 + it \quad (t \in [-1, 1])$

(c) $\gamma(t) = 1 - t + it \quad (t \in [0, 1])$

(d) $\gamma(t) = 2\cos t + 5i \sin t \quad (t \in [0, 2\pi])$

In each case determine the equation of Γ in terms of x and y.

It is important to realise that a given set can be considered as many different paths by using different parametrisations. For example, the functions

$$\gamma(t) = t \quad (t \in [0,1]) \quad \text{and} \quad \gamma(t) = t^2 \quad (t \in [0,1])$$

are both parametrisations of the real interval $[0, 1]$ in $\mathbb{C}$. As indicated in Figure 2.7, for the parametrisation $\gamma(t) = t$, the progress of the point along the interval $[0, 1]$ is uniform – for example, it is halfway along at time $t = \frac{1}{2}$. But for the parametrisation $\gamma(t) = t^2$, the speed of the point varies with t – for example, in the time interval $0 \leq t \leq \frac{1}{2}$, the point has travelled one-quarter of the distance along the interval $[0, 1]$, and in the time interval $\frac{1}{2} \leq t \leq 1$, it travels the next three-quarters. (In fact, in Unit A4, you will see that the speed of the point at time t is given by the modulus of the derivative of γ at t.)

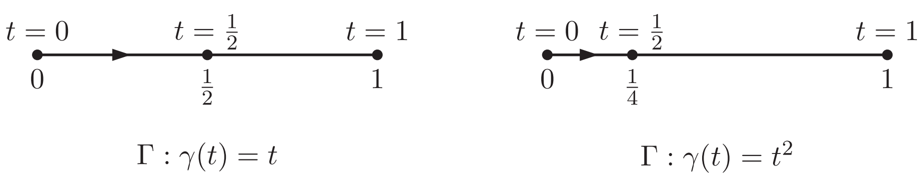
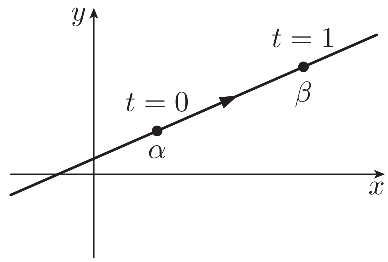
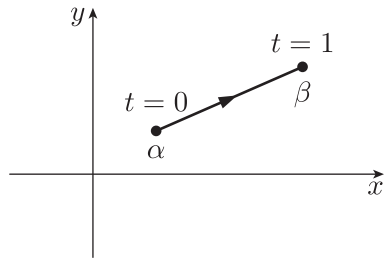
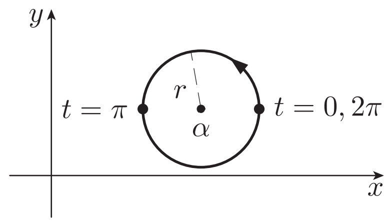
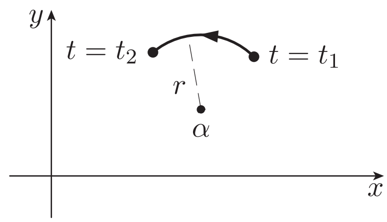
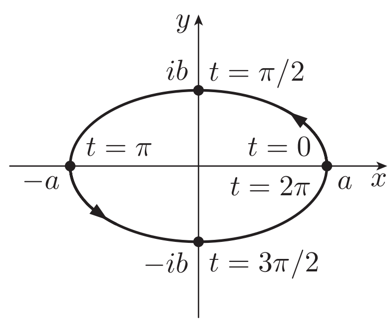


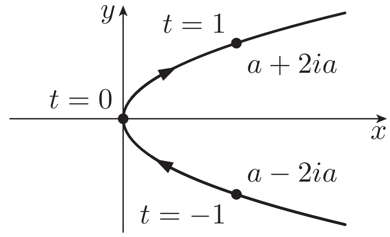
Figure 2.7 Two parametrisations of Γ

Various types of sets (such as line segments and arcs of circles) occur frequently in this module as paths. We will normally use a *standard parametrisation* for each of these, as indicated in the following table.

Set	Standard parametrisation	Diagram
Line through α and β	$\gamma(t) = (1-t)\alpha + t\beta \quad (t \in \mathbb{R})$	
Line segment from α to β	$\gamma(t) = (1-t)\alpha + t\beta \quad (t \in [0,1])$	
Circle with centre α, radius r: $\lvert z-\alpha\rvert = r$	$\gamma(t) = \alpha + r(\cos t + i\sin t) \quad (t \in [0,2\pi])$	
Arc of circle with centre α, radius r	$\gamma(t) = \alpha + r(\cos t + i\sin t) \quad (t \in [t_1,t_2])$	
Ellipse in standard form: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $a, b > 0$	$\gamma(t) = a\cos t + ib\sin t \quad (t \in [0,2\pi])$	

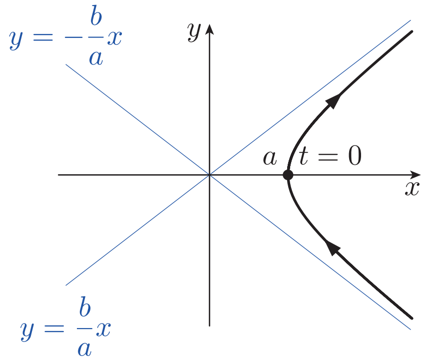
Parabola in standard form:
$y^2 = 4ax$,
where $a > 0$

$\gamma(t) = at^2 + 2iat \quad (t \in \mathbb{R})$



Right half of hyperbola in standard form:
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,
where $a, b > 0$

$\gamma(t) = a \cosh t + ib \sinh t \quad (t \in \mathbb{R})$



Exercise 2.4

For each of the following paths, write down the standard parametrisation and obtain the corresponding parametric equations:

(a) the line through -2 and i

(b) the line segment from 1 to $1 + i$

(c) the circle with centre $1 + i$ and radius 1

(d) the parabola $y^2 = x$.

Let us now consider how the image point $w = f(z)$ behaves as the point z moves around the domain of the function f. We make the following definition.

Definition

Given a function $f\colon A \longrightarrow B$ and a subset S of A, the **image under f of S**, written $f(S)$, is

$$f(S) = \{f(z) : z \in S\}.$$

Remarks

1. We often write 'f maps S to T' to mean $f(S) = T$.
2. If $S = A$, then, as noted earlier, $f(S)$ is also described as the image set of f.

Suppose that $f(z) = z^2$ and that the point z moves around the unit circle $|z| = 1$. In this case, our knowledge of the function f tells us that the image w of z satisfies $|w| = |z^2| = |z|^2 = 1$, and

$$\text{if } \theta \text{ is an argument of } z, \text{ then } 2\theta \text{ is an argument of } w$$

(see Figure 2.8). Thus if z moves once around the circle $|z| = 1$ anticlockwise, then w moves twice around the circle $|w| = 1$ anticlockwise.

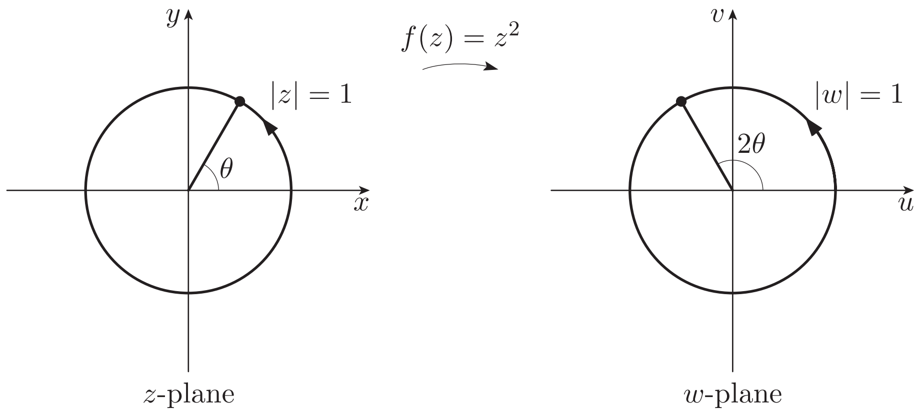


Figure 2.8 Image of the circle $|z| = 1$ under the function $f(z) = z^2$

This geometric observation can be made precise by using the parametrisation

$$\gamma(t) = \cos t + i \sin t \quad (t \in [0, 2\pi])$$

of the unit circle $|z| = 1$ traversed once anticlockwise. Now, from equations (2.1) in Subsection 2.1 we know that for $f(z) = z^2$, with $z = x + iy$ and $w = f(z) = u + iv$,

$$u = x^2 - y^2, \quad v = 2xy. \tag{2.2}$$

The parametric equations corresponding to the parametrisation γ are

$$x = \cos t, \quad y = \sin t \quad (t \in [0, 2\pi]), \tag{2.3}$$

and, on substituting these in equations (2.2), we obtain

$$u = \cos^2 t - \sin^2 t, \quad v = 2 \cos t \sin t;$$

that is,

$$u = \cos 2t, \quad v = \sin 2t \quad (t \in [0, 2\pi]). \tag{2.4}$$

An alternative way to obtain these equations is to observe that

$$u + iv = (x + iy)^2 = (\cos t + i \sin t)^2 = \cos 2t + i \sin 2t,$$

by De Moivre's Theorem, and then equate real and imaginary parts.

Equations (2.4) are the parametric equations for the image circle $|w| = 1$. From these equations we can verify our earlier assertion that as t increases from 0 to 2π, z moves *once* around the circle $|z| = 1$ anticlockwise while w moves *twice* around the circle $|w| = 1$ anticlockwise.

In the next definition we assume that a complex function f is 'continuous'. This is to ensure that if γ is the parametrisation of a path, then $f \circ \gamma$ is also the parametrisation of a path. Continuous functions and their properties are discussed in Unit A3.

Definition

Let f be a continuous function, and let Γ be a path in the domain of f. Then $f(\Gamma)$ is called the **image path** (under f of Γ). If Γ has parametrisation γ, then $f(\Gamma)$ has parametrisation $f \circ \gamma$, which is the function with rule $t \longmapsto f(\gamma(t))$.

Remark

If γ is the standard parametrisation of Γ, then $f \circ \gamma$ is certainly a parametrisation of $f(\Gamma)$, but it need not be the *standard* parametrisation of $f(\Gamma)$. For example, consider again the function $f(z) = z^2$, which maps the unit circle Γ to itself. The standard parametrisation of Γ is $\gamma(t) = \cos t + i \sin t$ $(t \in [0, 2\pi])$, but, as we have seen, $(f \circ \gamma)(t) = \cos 2t + i \sin 2t$ $(t \in [0, 2\pi])$ is a parametrisation of Γ, but not the standard one.

The two approaches for finding image paths that we applied to the function $f(z) = z^2$ and the unit circle are summarised in the following strategy.

Strategy for determining an image path

Let f be a continuous function, and let Γ be a path with parametrisation

$$\gamma(t) = \phi(t) + i\psi(t) \quad (t \in I).$$

To find the image path $f(\Gamma)$

- *either* use the geometric properties of f
- *or* substitute $x = \phi(t)$, $y = \psi(t)$ into the equation

  $$u + iv = f(x + iy),$$

  and then, by equating real parts and imaginary parts, obtain expressions for u and v in terms of t. (These expressions are the parametric equations of the image path $f(\Gamma)$ associated with the parametrisation $f \circ \gamma$.)

You will practise applying this strategy in the next section.

Further exercises

Exercise 2.5

Determine the real and imaginary parts, $\operatorname{Re} f$ and $\operatorname{Im} f$, of each of the following functions.

(a) $f(z) = \overline{z}$ (b) $f(z) = iz$ (c) $f(z) = z^3$ (d) $f(z) = |z|$

Exercise 2.6

Sketch each of the paths Γ with the following parametrisations.

(a) $\gamma(t) = 1 - it \quad (t \in \mathbb{R})$ (b) $\gamma(t) = i + (1-i)t \quad (t \in [0,1])$

(c) $\gamma(t) = \cos t - i \sin t \quad (t \in [0, 2\pi])$

Exercise 2.7

For each of the following parametrisations γ, find the equation of the corresponding path Γ in terms of x and y only. Sketch and classify the path in each case.

(a) $\gamma(t) = (1-t)(1+i) + ti \quad (t \in \mathbb{R})$

(b) $\gamma(t) = 2\cos t + 3i \sin t \quad (t \in [0, 2\pi])$

(c) $\gamma(t) = 1 + 2\cos t - (1 - 2\sin t)i \quad (t \in [0, 2\pi])$

Exercise 2.8

Determine the standard parametrisation for each of the following sets:

(a) the circle with centre $1 - i$ and radius 3

(b) the ellipse $2x^2 + 3y^2 = 6$

(c) the parabola $8y^2 = x$.

Exercise 2.9

Sketch the path with parametrisation

$$\gamma(t) = \tfrac{1}{2}(\cos t + i \sin t) - \tfrac{1}{4}(\cos 2t + i \sin 2t) \quad (t \in [-\pi, \pi])$$

by first plotting $\gamma(t)$ for $t = 0, \pm\dfrac{\pi}{4}, \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{4}, \pm\pi$.

Verify that the equation of the path is

$$4(x^2 + y^2)^2 - \tfrac{3}{2}(x^2 + y^2) + \tfrac{1}{2}x = \tfrac{3}{64}.$$

Exercise 2.10

Determine the image under the function $f(z) = \sqrt{z}$ of each of the following sets:

(a) the negative real axis (b) the circle $|z| = 1$.

3 Images of grids

After working through this section, you should be able to:

- sketch the image of a given Cartesian grid under various complex functions
- sketch the image of a given polar grid under various complex functions.

3.1 Cartesian and polar grids

To obtain a clear picture of the geometric effect of a given complex function, we can consider the images of many lines or paths in the domain of the function. In order to do this in a systematic way, we introduce two types of grid.

The first type of grid is a **Cartesian grid** consisting of lines of the form $x = a$ and $y = b$, usually evenly spaced in each direction (Figure 3.1(a)). The second type consists of circles with centre 0 and rays emerging from 0; it is called a **polar grid** because of the connection with the polar form $z = r(\cos\theta + i\sin\theta)$. Each of the circles has an equation of the form $r = a$, where a is a positive constant, and each of the rays has an equation of the form $\theta = b$, where b is a constant in the interval $(-\pi, \pi]$ (see Figure 3.1(b)).

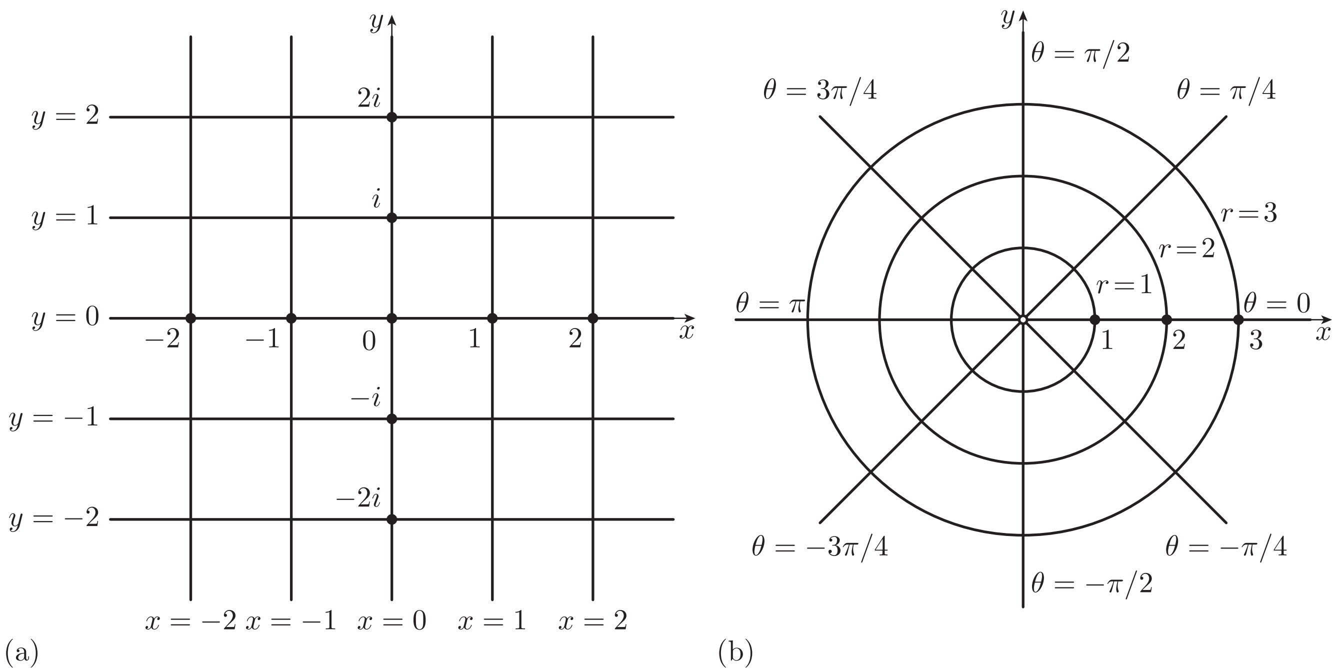


Figure 3.1 (a) A Cartesian grid (b) A polar grid

Exercise 3.1

Plot the polar grid consisting of the circles $r = 1$, $r = \frac{1}{2}$, $r = \frac{1}{3}$ and the rays $\theta = 0$, $\theta = \pm\pi/3$, $\theta = \pm 2\pi/3$, $\theta = \pi$.

3.2 Images of Cartesian and polar grids

In this subsection we examine the images of Cartesian and polar grids under three different complex functions to give us insight into the geometric effects of these functions. The images are found by using the strategy for determining an image path given at the end of Subsection 2.2.

In each case we have also highlighted (by using shading) the effect of the function on a particular set bounded by parts of the grid.

Before we consider these examples, you should try the following exercise, the results of which will be used in the examples to follow.

Exercise 3.2

Eliminate t from each of the following pairs of parametric equations. (In each case, a is a real constant, with $a \neq 0$ in parts (b) and (c).)

(a) $u = a - t, \quad v = a + t$ (b) $u = a^2 - t^2, \quad v = 2at$

(c) $u = \dfrac{a}{a^2 + t^2}, \quad v = \dfrac{-t}{a^2 + t^2}$

Images of grids under $f(z) = (1 + i)z$

The first function we consider is the linear function $f(z) = (1 + i)z$. For convenience we define $w = f(z)$. According to the strategy for determining an image path, one method for finding images of sets under f involves splitting both z and $w = f(z)$ into real and imaginary parts, say $z = x + iy$ and $w = u + iv$. The equation $w = (1 + i)z$ then becomes

$$u + iv = (1 + i)(x + iy) = (x - y) + i(x + y).$$

Equating real and imaginary parts gives

$$u = x - y, \quad v = x + y. \tag{3.1}$$

We wish to find the image of a Cartesian grid under f, and to this end let us first work out the image of a vertical line $x = a$, where a is a real constant. It is convenient to think of this line as a path, and represent it by the parametric equations

$$x = a, \quad y = t \quad (t \in \mathbb{R}),$$

so that the line now has a direction: 'pointing upwards'. Substituting the expressions for x and y into equations (3.1) gives

$$u = a - t, \quad v = a + t.$$

We can now eliminate t from these equations, as you were asked to do in Exercise 3.2(a), to obtain

$$u + v = 2a,$$

which is the equation of a line in the w-plane.

The line $x = a$ and its image line $u + v = 2a$ are illustrated in Figure 3.2.

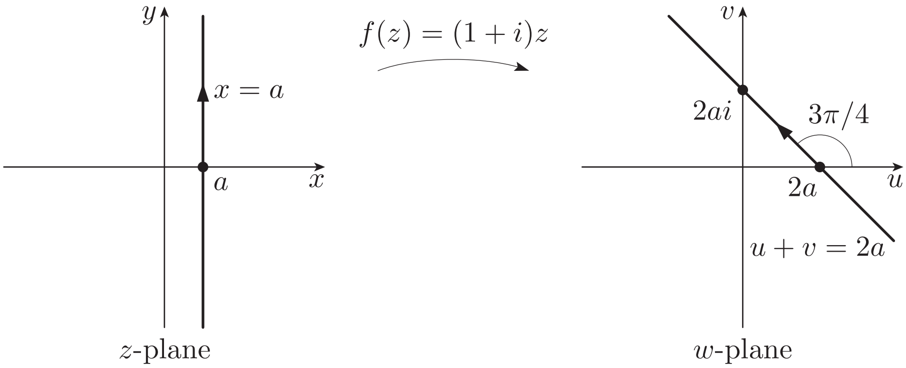


Figure 3.2 Image of a vertical line under $f(z) = (1+i)z$

The next exercise asks you to find the images of horizontal lines under f.

Exercise 3.3

Determine the image of the line $y = b$, where $b \in \mathbb{R}$, under the function $f(z) = (1+i)z$.

Sketch the images of the lines $y = 1$ and $y = 0$.

Now that we have worked out the images of horizontal and vertical lines under f, we can find the image of a Cartesian grid under f. The result is illustrated in Figure 3.3.

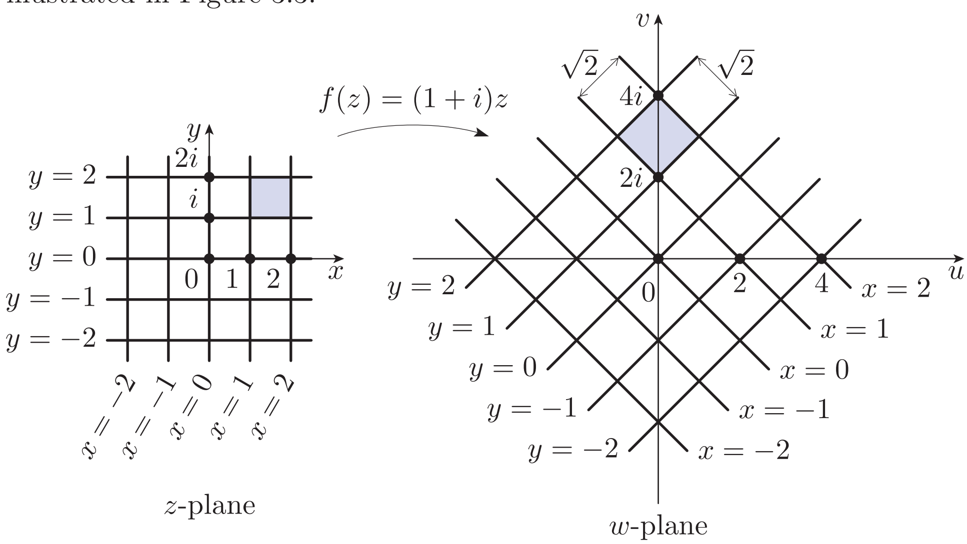


Figure 3.3 Image of a Cartesian grid under $f(z) = (1+i)z$

The vertical lines in the grid in the z-plane all have equations of the form $x = a$, for various values of a. As we have seen, these lines are mapped by f to lines that slope from top-left to bottom-right (negative gradient) in the w-plane. The horizontal lines in the z-plane have equations $y = b$, for various values of b, and these are mapped by f to lines that slope from bottom-left to top-right (positive gradient) in the w-plane.

Each line in the w-plane in Figure 3.3 is labelled by the equation of the corresponding line in the z-plane to help you appreciate how the grid is transformed. We continue this convention whenever we sketch images of grids.

By studying Figure 3.3, you can see that the image of the grid in the z-plane is another grid in the w-plane, but one that is rotated anticlockwise about the origin by $\pi/4$. Furthermore, the grid is scaled by the factor $\sqrt{2}$, as shown in Figure 3.3. The figure illustrates the effect of the transformation on a shaded square, which is scaled and rotated under f.

Another way to understand the behaviour of f is to write $1+i$ in polar form as

$$1+i = \sqrt{2}(\cos\pi/4 + i\sin\pi/4).$$

The geometric effect of multiplying z by $1+i$, which is what f does, is to scale z by the modulus of $1+i$ (namely $\sqrt{2}$) and rotate z anticlockwise about the origin by the principal argument of $1+i$ (namely $\pi/4$).

Figure 3.4 shows the image of a polar grid under the same function f. Each circle $r=a$ in the z-plane, where a is a positive constant, is scaled by a factor $\sqrt{2}$, and each ray $\theta=b$, where $b\in(-\pi,\pi]$, is rotated anticlockwise by $\pi/4$.

Once again, each curve in the w-plane is labelled by the equation of the corresponding curve in the z-plane.

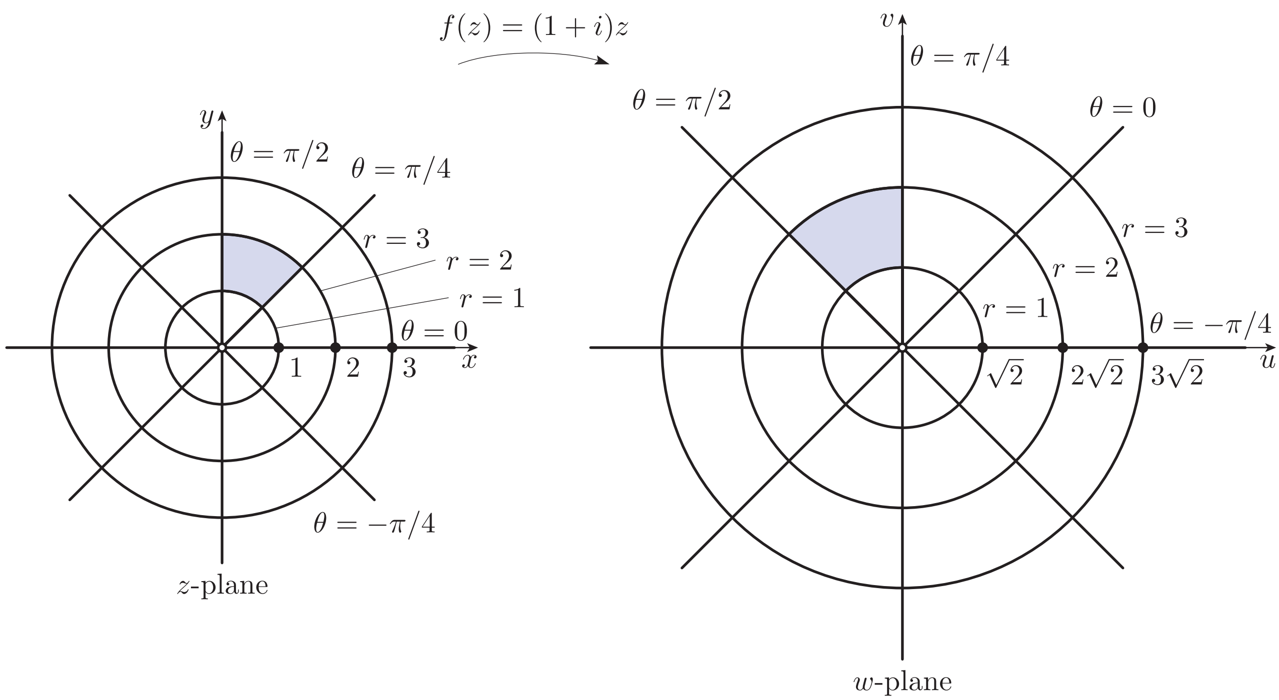


Figure 3.4 Image of a polar grid under $f(z) = (1+i)z$

Images of grids under $f(z) = z^2$

Our next example is the function $f(z) = z^2$. As before, we let $w = f(z)$ and write $z = x + iy$ and $w = u + iv$. From equations (2.1) in Subsection 2.1, we know that

$$u = x^2 - y^2, \quad v = 2xy. \tag{3.2}$$

Let us use these equations to find the image under f of the vertical line $x = a$, where a is a positive constant. The line $x = a$ has parametric equations

$$x = a, \quad y = t \quad (t \in \mathbb{R}),$$

so the parametric equations of the image of $x = a$ are

$$u = a^2 - t^2, \quad v = 2at \quad (t \in \mathbb{R}).$$

In Exercise 3.2(b) you were asked to eliminate t from this pair of equations, to obtain

$$v^2 = 4a^2(a^2 - u).$$

This is the equation of a parabola, provided that $a \neq 0$, as illustrated in Figure 3.5.

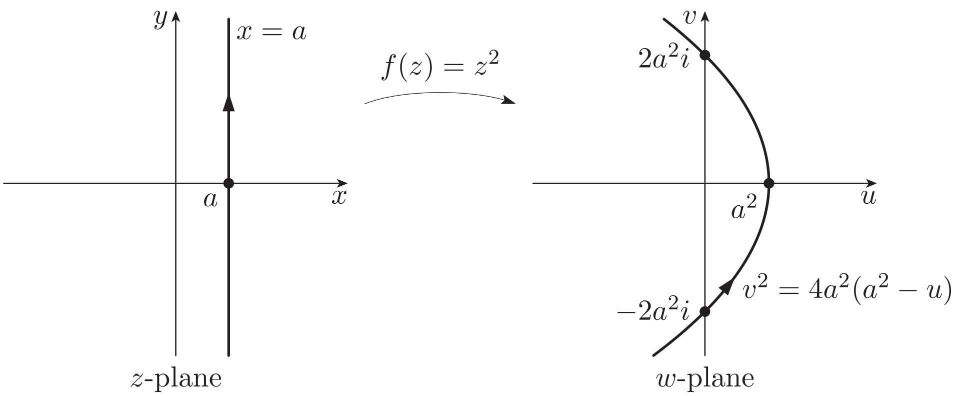


Figure 3.5 Image of a vertical line under $f(z) = z^2$

In the exceptional case when $a = 0$, we obtain the parametric equations $u = -t^2$ and $v = 0$, so the image of the vertical line in this case is the negative real axis together with the origin. As t increases, the point $u + iv$ moves along the negative real axis in the w-plane, first from left to right (until it reaches 0), and then back again. This is illustrated in Figure 3.6.

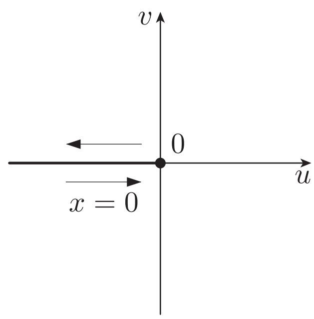


Figure 3.6 Image of the line $x = 0$ under $f(z) = z^2$

Now try the next exercise, which asks you to find the images of horizontal lines under f.

Exercise 3.4

Determine the image of the line $y = b$, where $b \in \mathbb{R}$, under the function $f(z) = z^2$.

Sketch the images of the lines $y = 1$ and $y = 0$.

Figure 3.7 shows the image of a Cartesian grid under f. Notice that for $a \neq 0$, the vertical lines $x = a$ and $x = -a$ both map to the same parabola $v^2 = 4a^2(a^2 - u)$, because $(-a)^2 = a^2$. Likewise, for $b \neq 0$, the horizontal lines $y = b$ and $y = -b$ both map to the parabola $v^2 = 4b^2(u + b)$.

We can see the effect of the function on the shaded square in the z-plane: it is mapped to the shaded set in the w-plane bounded by parts of four parabolas.

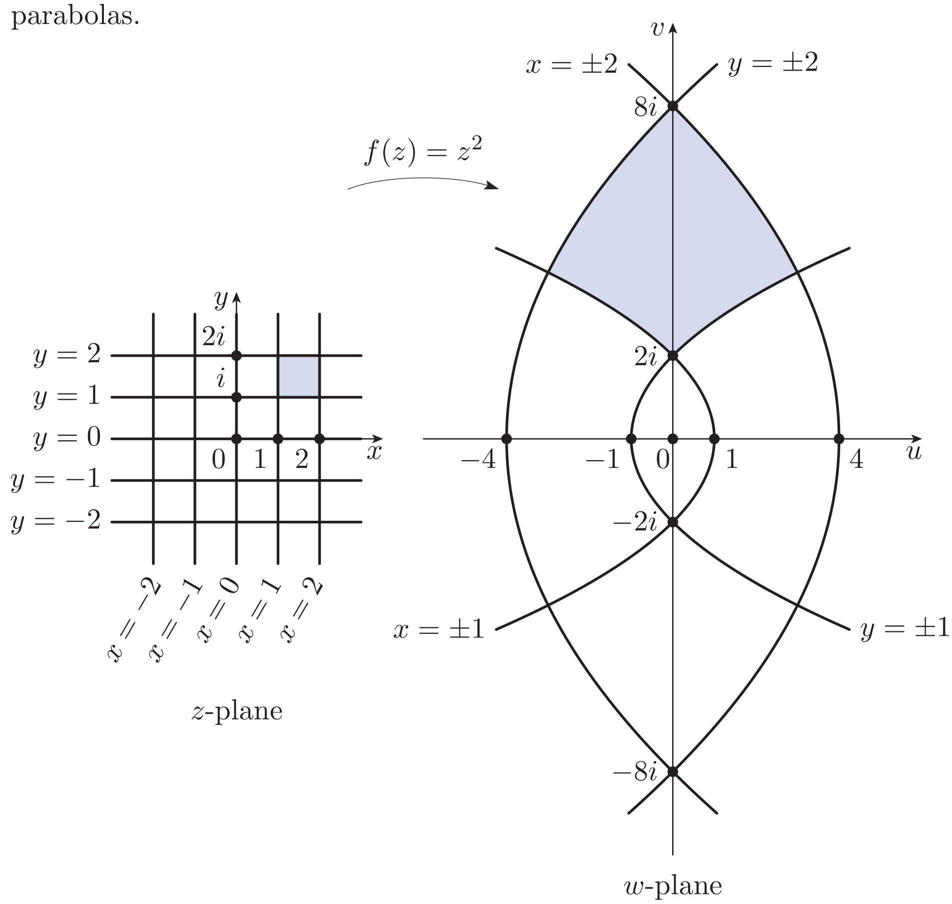


Figure 3.7 Image of a Cartesian grid under $f(z) = z^2$

As with the previous example, we can better understand the geometric effect of f by writing z in polar form as $z = r(\cos\theta + i\sin\theta)$, so

$$w = z^2 = r^2(\cos 2\theta + i\sin 2\theta).$$

From this formula we see that f squares the modulus of each complex number and doubles the argument. These properties are demonstrated using a polar grid in Figure 3.8. The radii of the concentric circles in the z-plane are all squared, and the arguments of the rays are all doubled. For example, the ray in the z-plane with argument $\pi/4$ is mapped to the ray in the w-plane with argument $\pi/2$, the positive imaginary axis. In fact, because arguments are doubled, any ray in the w-plane is the image of precisely two rays in the z-plane. For instance, the positive imaginary axis is also the image of the ray with argument $-3\pi/4$, because doubling $-3\pi/4$ gives $-3\pi/2 = \pi/2 - 2\pi$.

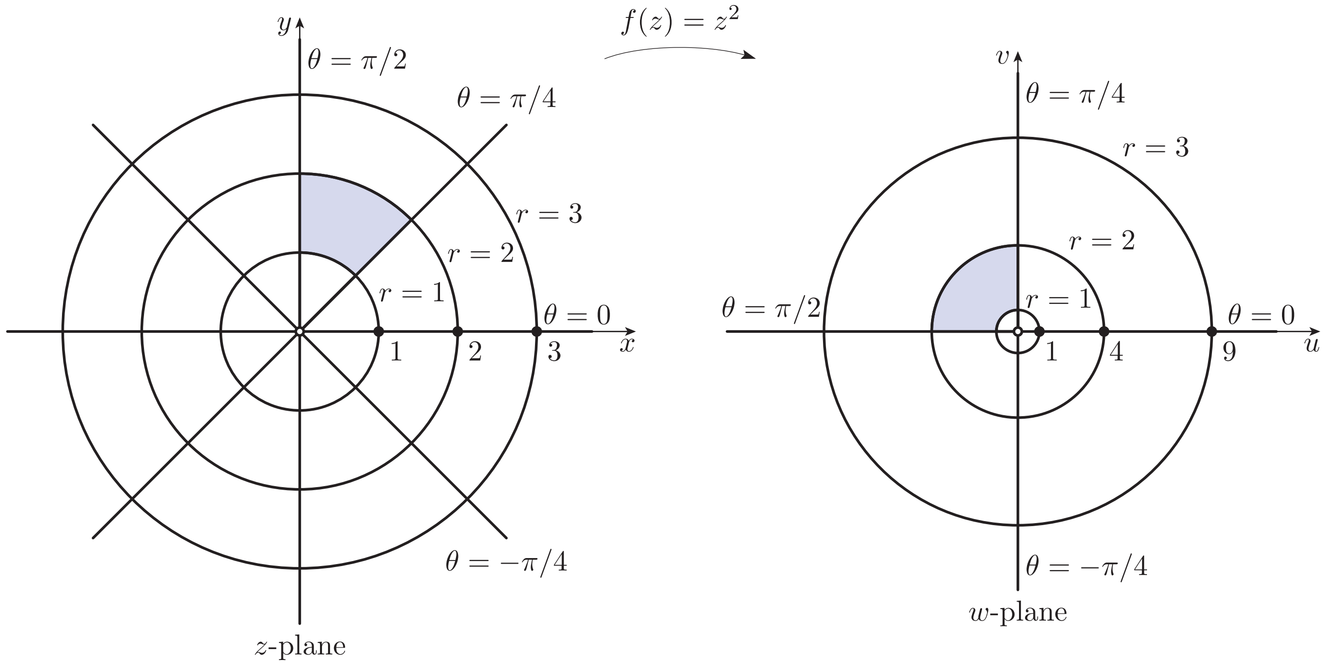


Figure 3.8 Image of a polar grid under $f(z) = z^2$

In Figure 3.8 and some other similar figures later in the unit we have used different scales for the z-plane and w-plane in order to display the features of the grids clearly.

Images of grids under $f(z) = 1/z$

Our last example is the function $f(z) = 1/z$, which has domain $\mathbb{C} - \{0\}$. It is useful to observe that if we multiply the top and bottom of $1/z$ by $\overline{z}$, then we obtain another expression for $f(z)$, namely $f(z) = \overline{z}/|z|^2$. Next, if we let $w = f(z)$, and write $z = x + iy$ and $w = u + iv$ as usual, then we see that

$$u + iv = \frac{\overline{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2}.$$

Equating real and imaginary parts gives

$$u = \frac{x}{x^2 + y^2}, \quad v = -\frac{y}{x^2 + y^2}. \tag{3.3}$$

We can now find the parametric equations of the image of a vertical line $x = a$, where a is a non-zero real constant. The parametric equations of the line $x = a$ are

$$x = a, \quad y = t \quad (t \in \mathbb{R}),$$

so the parametric equations of the image of $x = a$ are

$$u = \frac{a}{a^2 + t^2}, \quad v = -\frac{t}{a^2 + t^2} \quad (t \in \mathbb{R}).$$

Eliminating t from these equations, as in Exercise 3.2(c), gives

$$u^2 + v^2 = u/a.$$

We can rewrite this equation as

$$(u - 1/(2a))^2 + v^2 = 1/(2a)^2,$$

which is the equation of a circle with centre $1/(2a)$ and radius $1/(2|a|)$. This circle passes through the origin and the point $1/a$, as shown in Figure 3.9. As t increases from $-\infty$ to ∞, the image point $u + iv$ moves clockwise around the circle, starting and finishing at the origin. However, the origin itself is excluded from the image, because as t approaches $-\infty$ or ∞, the point $u + iv$ on the circle gets closer and closer to the origin without actually reaching it.

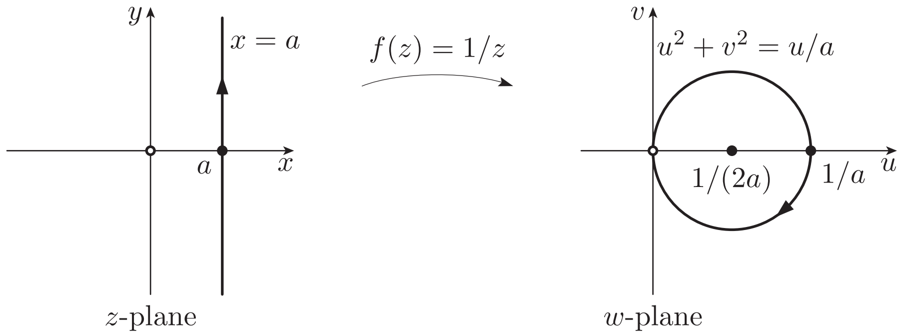


Figure 3.9 Image of a vertical line under $f(z) = 1/z$

It remains to consider the image of the line $x = a$, when $a = 0$. This line is split into two parts, the positive and negative imaginary axes, by the point 0, which is excluded from the domain of f. You can check that f maps the positive imaginary axis in the z-plane to the negative imaginary axis in the w-plane, and it maps the negative imaginary axis in the z-plane to the positive imaginary axis in the w-plane. The image of the line $x = 0$ under f is shown in Figure 3.10.

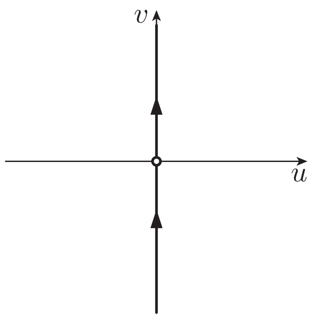


Figure 3.10 Image of the line $x = 0$ under $f(z) = 1/z$

In the following exercise, you are asked to find the images of horizontal lines under f.

Exercise 3.5

Determine the image of the line $y = b$, where $b \in \mathbb{R}$, under the function $f(z) = 1/z$.

Sketch the images of the lines $y = 1$ and $y = 0$.

Figure 3.11 shows the image of a Cartesian grid under $f(z) = 1/z$, with different scales for the z-plane and the w-plane. The shaded square in the z-plane is mapped to a set in the w-plane bounded by parts of four circles.

Once again, we can gain insight into the geometric effect of f by writing z in polar form as $z = r(\cos\theta + i\sin\theta)$, so

$$w = \frac{1}{z} = \frac{1}{r}(\cos(-\theta) + i\sin(-\theta)).$$

This formula shows that f changes the modulus of z to its reciprocal, and reverses the sign of the argument of z. This is demonstrated by the transformation of the polar grid in Figure 3.12.

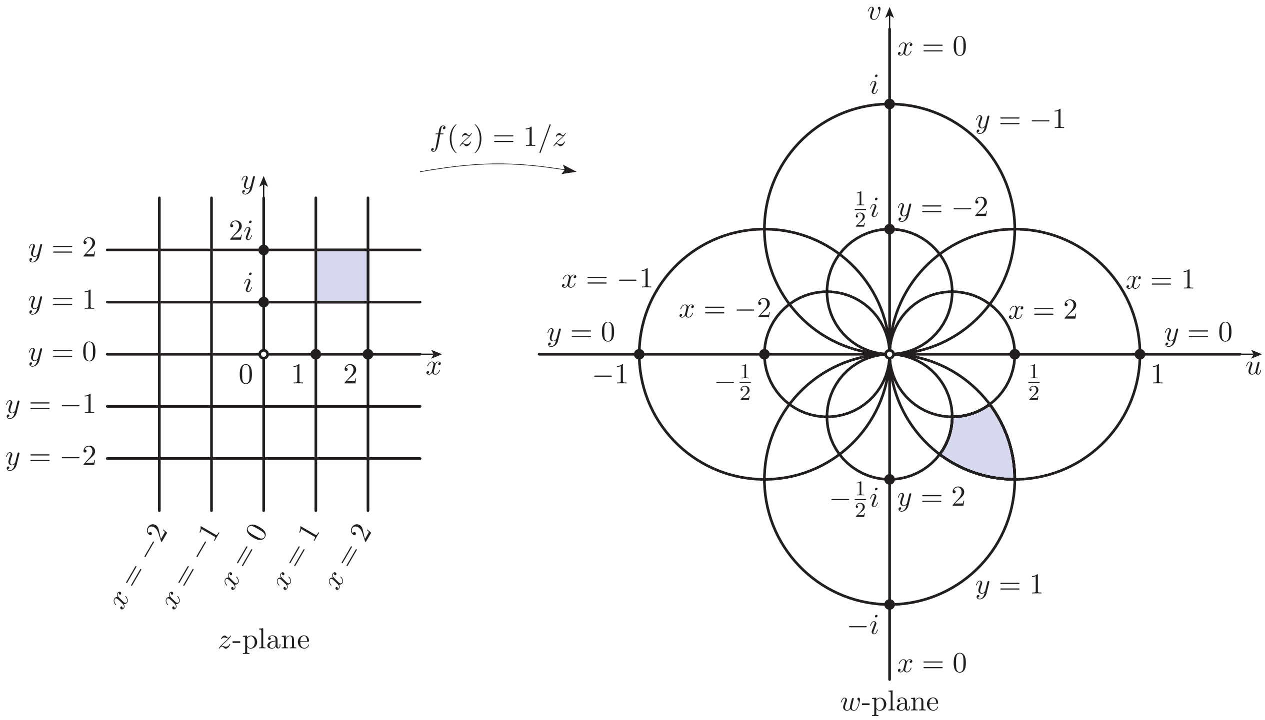


Figure 3.11 Image of a Cartesian grid under $f(z) = 1/z$

In Figure 3.12 a circle centred at the origin in the z-plane of radius a is mapped by f to a circle centred at the origin in the w-plane of radius $1/a$, and the ray with argument b is mapped to the ray with argument $-b$.

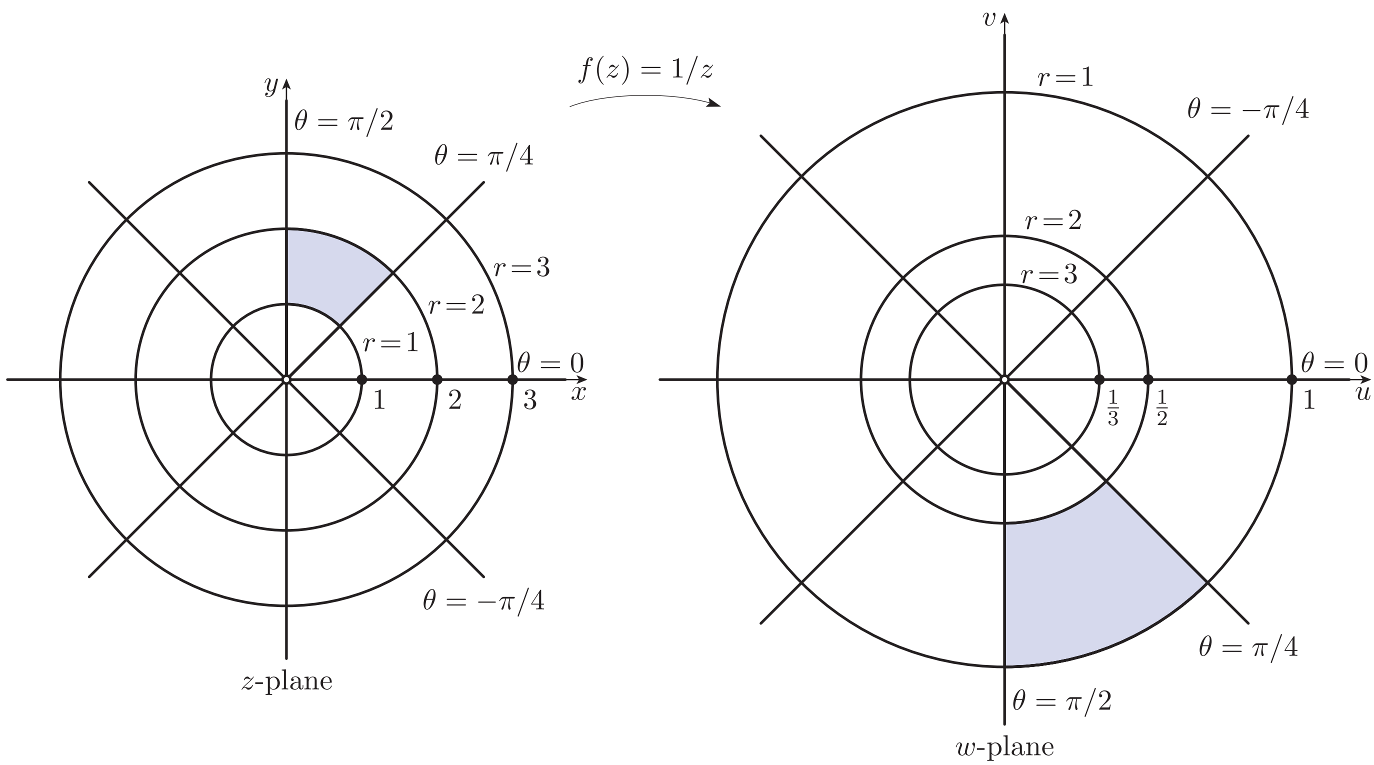


Figure 3.12 Image of a polar grid under $f(z) = 1/z$

The following exercises provide some practice in determining images of Cartesian grids and polar grids for complex functions. In such exercises, your geometric knowledge of the function f may save you from getting involved in parametrisations.

Exercise 3.6

For the function $f(z) = iz + 1$, sketch the images of

(a) $S = \{z : 0 \leq \operatorname{Re} z \leq 1,\ 0 \leq \operatorname{Im} z \leq 1\}$

(b) the polar grid of Figure 3.1(b).

Exercise 3.7

Sketch the image of the polar grid of Figure 3.1(b) under the function $f(z) = z^3$, omitting the circle with equation $r = 3$. (Its image is rather large.)

Exercise 3.8

Find the image of the polar grid of Figure 3.1(b) under the function

$$f(z) = \sqrt{z}.$$

Further exercises

Exercise 3.9

Sketch the images under each of the following functions f of the Cartesian grid and polar grid shown in Figure 3.1.

(a) $f(z) = z + i$ (b) $f(z) = 2z$ (c) $f(z) = 2 - iz$

(d) $f(z) = iz^2$

Vector fields

In this unit we have seen how to illustrate complex functions using two copies of the complex plane, one representing the domain of a function and the other representing the codomain. There is an alternative method for representing a complex function f geometrically using only a single copy of the complex plane. In this method, the value $f(z)$ is marked by an arrow emanating from the point z that has magnitude $|f(z)|$ and that makes an angle $\operatorname{Arg} f(z)$ with the positive horizontal direction. The resulting collection of points and arrows is called the *vector field* of f.

For instance, Figure 3.13 illustrates the vector field of the function $f(z) = z^2$.

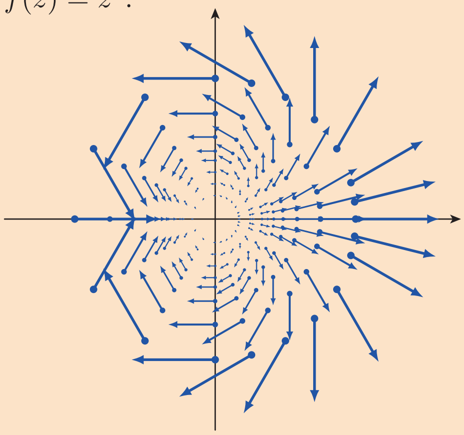


Figure 3.13 Vector field of $f(z) = z^2$

Vector fields are widely used in science to represent physical phenomena. For example, the arrows in Figure 3.13 might represent the velocities of particles in a flowing fluid. The particles on the positive real axis are propelled rightwards along the positive real axis. Other particles follow trajectories that guide them towards the origin. The arrows shrink as they approach the origin, indicating that the origin is fixed by the function. You will learn about fluid flows in Unit D1.

4 Exponential, trigonometric and hyperbolic functions

After working through this section, you should be able to:

- state and use the definition and basic algebraic properties of the exponential function and describe its geometric properties
- state and use the definitions and basic algebraic properties of the trigonometric functions and hyperbolic functions
- prove simple identities involving the exponential function, the trigonometric functions and the hyperbolic functions.

4.1 The exponential function

The real exponential function $x \longmapsto e^x$, which is illustrated by the graph in Figure 4.1, plays a key role in a wide range of mathematical subjects. It is natural to ask whether this function has a complex analogue; that is, how is e^z defined where z is a complex number?

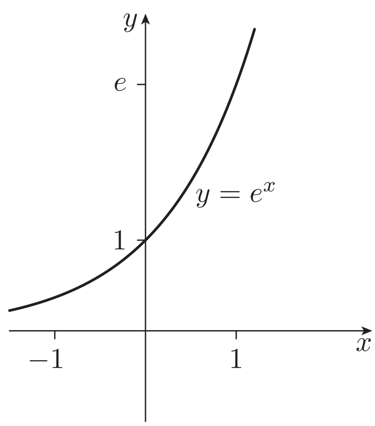


Figure 4.1 Graph of $y = e^x$

A fundamental property of the real exponential function is that

$$e^{x_1}e^{x_2} = e^{x_1+x_2}, \quad \text{for all } x_1, x_2 \in \mathbb{R}.$$

Ideally, then, the complex exponential function should satisfy

$$e^{z_1}e^{z_2} = e^{z_1+z_2}, \quad \text{for all } z_1, z_2 \in \mathbb{C}. \tag{4.1}$$

In particular, if $z = x + iy$, then it should be true that

$$e^z = e^{x+iy} = e^x e^{iy}.$$

Since e^x *is* defined, because x is real, it remains only to define e^{iy}, where y is real. The following manipulation of power series suggests a definition of e^{iy}. The power series for e^x, where $x \in \mathbb{R}$, is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

Let us now try replacing x with iy in this formula. We cannot justify doing so at this stage because iy is not a real number; nonetheless, the outcome is illuminating.

We obtain

$$\begin{aligned} e^{iy} &= 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \cdots \\ &= \left(1 - \frac{y^2}{2!} + \cdots\right) + i\left(y - \frac{y^3}{3!} + \cdots\right). \end{aligned}$$

The expressions in parentheses are the power series for $\cos y$ and $\sin y$, respectively. Thus it seems plausible to define e^{iy} by

$$e^{iy} = \cos y + i \sin y;$$

this formula is known as **Euler's Identity** (or 'Euler's Formula'), named after the Swiss mathematician Leonhard Euler (pronounced 'oiler'), whom we met in Unit A1 and whom we will meet again later.

Definition

For all $z = x + iy$ in $\mathbb{C}$,

$$e^z = e^x(\cos y + i \sin y).$$

The function

$$z \longmapsto e^z \quad (z \in \mathbb{C})$$

is called the **exponential function**, and is denoted by exp. Thus $\exp z = e^z$.

Before verifying that the exponential function does indeed satisfy equation (4.1), we make some remarks about the definition and then evaluate e^z for several complex numbers z.

Remarks

1. Notice that if z is real, so $z = x + 0i$ (that is, $z = x$), then

   $$e^z = e^x(\cos 0 + i \sin 0) = e^x.$$

   Thus the restriction of the (complex) exponential function to $\mathbb{R}$ gives the real exponential function, as we would expect. In particular, for $0 \in \mathbb{C}$,

   $$e^0 = 1.$$

2. Also, if $z = 0 + iy$, then

   $$e^z = e^0(\cos y + i \sin y),$$

   which gives Euler's Identity

   $$e^{iy} = \cos y + i \sin y.$$

   Thus the number e^{iy} has modulus 1 and argument y, and so lies on the unit circle $\{z : |z| = 1\}$. Some important examples are given in Figure 4.2.

   In particular, $e^{i\pi} = -1$; that is,

   $$e^{i\pi} + 1 = 0.$$

   This striking equation contains five of the most important numbers in mathematics, together with two of the most important symbols. It is known as 'Euler's Equation', or even sometimes 'Euler's Identity', although it is not to be confused with the more general formula $e^{iy} = \cos y + i \sin y$ that we call Euler's Identity.

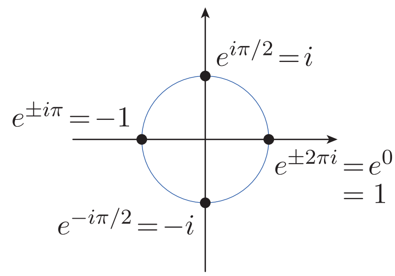


Figure 4.2 Some values of the exponential function on the unit circle

Example 4.1

Express each of the following numbers in Cartesian form.

(a) $e^{i\pi/3}$ (b) $e^{(1-i\pi)/2}$ (c) $e^{-1+i\pi/4}$

Solution

(a) $e^{i\pi/3} = e^0(\cos \pi/3 + i \sin \pi/3) = 1/2 + i\sqrt{3}/2$

(b) $e^{(1-i\pi)/2} = e^{1/2}(\cos(-\pi/2) + i \sin(-\pi/2)) = -ie^{1/2}$

(c) $$\begin{aligned} e^{-1+i\pi/4} &= e^{-1}(\cos \pi/4 + i \sin \pi/4) \\ &= e^{-1}(1/\sqrt{2} + i/\sqrt{2}) = \frac{1+i}{e\sqrt{2}} \end{aligned}$$

Exercise 4.1

Express each of the following numbers in Cartesian form.

(a) $e^{2\pi i}$ (b) $e^{2+i\pi/3}$ (c) $e^{-(1+i\pi)}$

The next result gives a number of basic identities involving the exponential function, including equation (4.1). Here and subsequently we adopt the convention that, unless otherwise stated, identities hold for all values of the variables in the identity for which the identity has meaning. For example, the first identity below holds for all z_1 and z_2 in $\mathbb{C}$.

Theorem 4.1 Exponential Identities

(a) **Addition** $e^{z_1+z_2} = e^{z_1}e^{z_2}$

(b) **Modulus** $|e^z| = e^{\operatorname{Re} z}$

(c) **Negatives** $e^{-z} = 1/e^z$

(d) **Periodicity** $e^{z+2\pi i} = e^z$

Remarks

1. One consequence of part (a) is that
$$(e^{i\theta})^n = e^{in\theta}, \quad \text{for } \theta \in \mathbb{R},\ n \in \mathbb{Z}. \tag{4.2}$$
This is a restatement of De Moivre's Theorem (Theorem 2.2 of Unit A1) in a concise form.
2. Since the real exponential function is always positive, part (b) shows that $e^z \neq 0$ for all $z \in \mathbb{C}$.
3. Part (d) shows that the exponential function is *periodic* with *period* $2\pi i$, meaning that exp takes the same value at z and $z + 2\pi i$, for any complex number z.

Proof We prove parts (a), (b) and (c), leaving part (d) as an exercise.

(a) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then
$$\begin{aligned} e^{z_1}e^{z_2} &= e^{x_1}(\cos y_1 + i\sin y_1)e^{x_2}(\cos y_2 + i\sin y_2)\\ &= e^{x_1}e^{x_2}(\cos y_1 + i\sin y_1)(\cos y_2 + i\sin y_2)\\ &= e^{x_1+x_2}((\cos y_1\cos y_2 - \sin y_1\sin y_2)\\ &\qquad\qquad + i(\sin y_1\cos y_2 + \cos y_1\sin y_2))\\ &= e^{x_1+x_2}(\cos(y_1+y_2) + i\sin(y_1+y_2))\\ &= e^{x_1+x_2}e^{i(y_1+y_2)}\\ &= e^{z_1+z_2}, \end{aligned}$$
since $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$.

(b) Let $z = x + iy$. Then
$$\begin{aligned} |e^z| &= |e^x(\cos y + i\sin y)|\\ &= |e^x||\cos y + i\sin y|\\ &= e^x, \end{aligned}$$
since $e^x > 0$ and $\cos^2 y + \sin^2 y = 1$. Hence $|e^z| = e^{\operatorname{Re} z}$.

(c) By part (a),
$$e^z e^{-z} = e^{z-z} = e^0 = 1.$$
Hence $e^{-z} = 1/e^z$. ∎

Exercise 4.2

(a) Prove part (d) of Theorem 4.1.

(b) Prove that

$$|e^z| \le e^{|z|}, \quad \text{for all } z \in \mathbb{C}.$$

(c) Is the function exp one-to-one?

(d) Determine each of the following sets.

(i) $\{z : e^z = 1\}$ (ii) $\{z : e^z = -1\}$

The result of Exercise 4.2(b) will prove to be useful later in the module.

The exponential function provides an alternative, concise notation for expressing a non-zero complex number in polar form.

Given a non-zero complex number z with modulus r and argument θ, both

$$z = r(\cos\theta + i\sin\theta) \quad \text{and} \quad z = re^{i\theta}$$

are acceptable ways of writing z in polar form.

For example,

$$1 + i = \sqrt{2}(\cos\pi/4 + i\sin\pi/4) = \sqrt{2}e^{i\pi/4}.$$

Using this concise version of polar form, it is easy to compute powers such as $(1+i)^{10}$ as follows:

$$\begin{aligned}
(1+i)^{10} &= (\sqrt{2}e^{i\pi/4})^{10} \\
&= (\sqrt{2})^{10}(e^{i\pi/4})^{10} \\
&= 32e^{10i\pi/4} && \text{(by equation (4.2))} \\
&= 32e^{5i\pi/2} \\
&= 32e^{i\pi/2} && \text{(exp has period } 2\pi i\text{)} \\
&= 32i.
\end{aligned}$$

Compare this calculation with that of Exercise 2.12(c) of Unit A1.

Exercise 4.3

By writing $\sqrt{3} + i$ in polar form, using the exponential function, evaluate $(\sqrt{3} + i)^{-6}$.

The geometric effect of the exponential function

Theorem 4.1(d) shows that the (complex) exponential function is not one-to-one, since

$$e^{z+2\pi i} = e^z \qquad (4.3)$$

but $z + 2\pi i \neq z$. Repeated application of equation (4.3) gives the following fact.

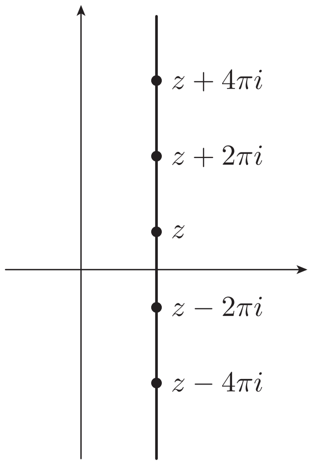


Figure 4.3 Points $z + 2n\pi i$ on the vertical line through z

$$e^{z+2n\pi i} = e^z \quad (n \in \mathbb{Z}).$$

Therefore each of the points

$$z + 2n\pi i, \quad n \in \mathbb{Z},$$

has the same image under the exponential function. These points lie on the vertical line through z, as shown in Figure 4.3. We will use this observation to investigate geometric properties of the exponential function.

As in Section 3, the aim is to plot the image of a grid of lines of the form $x = a$ and $y = b$, for suitable real constants a and b. To do this, let $w = e^z$, where $z = x + iy$ and $w = u + iv$, so

$$u + iv = e^z = e^x(\cos y + i \sin y).$$

Hence

$$u = e^x \cos y, \quad v = e^x \sin y. \qquad (4.4)$$

First we consider the image of a line of the form $x = a$, for a real constant a, with parametrisation $\gamma(t) = a + it$ $(t \in \mathbb{R})$. Substituting $x = a$ and $y = t$ into equations (4.4), we obtain the following property.

The function exp maps the line $x = a$ to the path with parametric equations

$$u = e^a \cos t, \quad v = e^a \sin t \quad (t \in \mathbb{R}).$$

This is the circle with centre 0 and radius e^a (Figure 4.4).

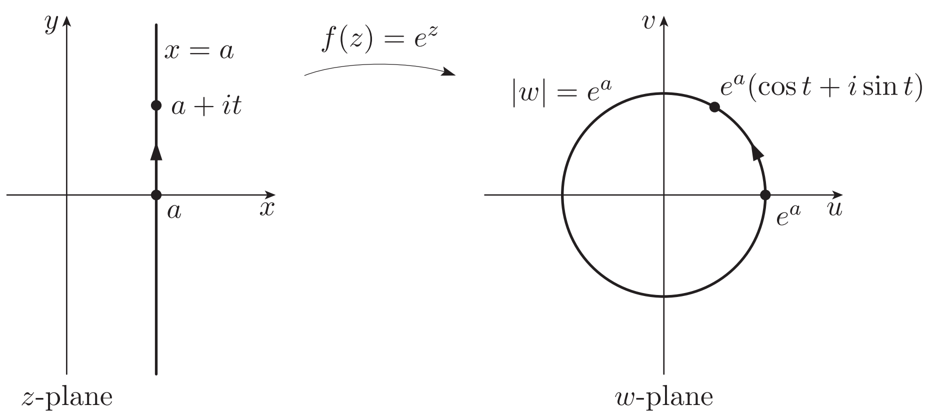


Figure 4.4 The line $\{x + iy : x = a\}$ and its image circle $\{w : |w| = e^a\}$

Notice that

- as t increases, the image point w moves anticlockwise around the image circle, passing through e^a whenever t is an integer multiple of 2π
- the image of the line $\{x + iy : x = 0\}$ is the unit circle $\{w : |w| = 1\}$
- as a increases, the image circle of the line $x = a$ expands, the centre remaining fixed at 0.

Next consider the image of a line of the form $y = b$, for a real constant b, with parametrisation $\gamma(t) = t + ib$ $(t \in \mathbb{R})$. Substituting $x = t$ and $y = b$ into equations (4.4), we obtain the following property.

The function exp maps the line $y = b$ to the path with parametric equations

$$u = e^t \cos b, \quad v = e^t \sin b \quad (t \in \mathbb{R}).$$

This is the ray from 0 (excluded) through $\cos b + i \sin b$ (Figure 4.5).

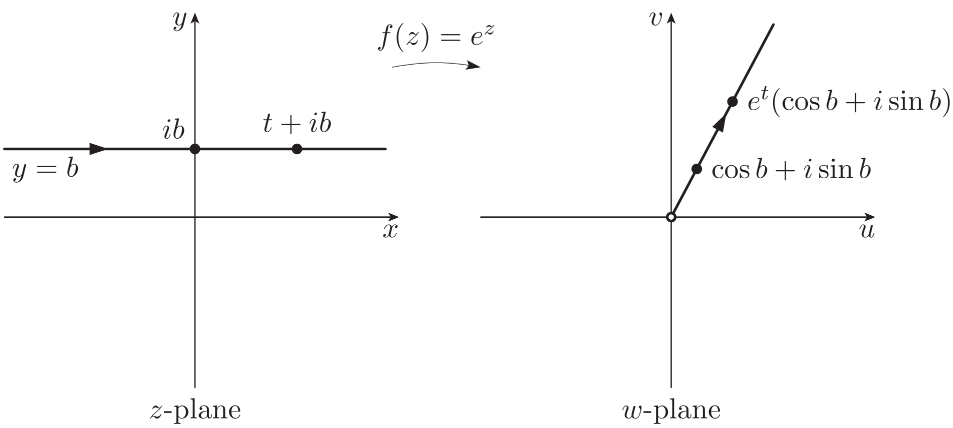


Figure 4.5 The line $\{x + iy : y = b\}$ and its image ray $\{w = \rho e^{i\phi} : \phi = b,\ \rho > 0\}$

Notice that

- as t increases, the image point w moves outwards along the image ray
- the image of the line $y = 0$ is the positive real axis
- as b increases, the image ray of the line $y = b$ rotates anticlockwise about 0.

Combining these observations, we can now plot the image of a grid of lines of the form $x = a$ and $y = b$. For our grid, we choose the values of a to be integers (as usual) but, because trigonometric functions are involved, it is convenient to choose the values of b to be integer multiples of $\pi/2$.

In Figure 4.6 the image circles of the lines $x = -2, -1, 0, 1, 2$ are shown, as are the image rays of the lines $y = -3\pi/2, -\pi, -\pi/2, 0, \pi/2, \pi$. (Note that the lines $y = -\pi$ and $y = \pi$, for example, have the same image.)

One effect of this choice of values for b is that the image of each grid rectangle in the z-plane is a quarter-annulus in the w-plane. In particular, notice that the two shaded rectangles in Figure 4.6 map to the same quarter-annulus.

Notice also that, since $|e^z| = e^{\operatorname{Re} z}$, points in the right half-plane $\{z : \operatorname{Re} z > 0\}$ have images lying outside the circle $\{w : |w| = 1\}$, whereas points in the left half-plane $\{z : \operatorname{Re} z < 0\}$ have images lying inside this circle.

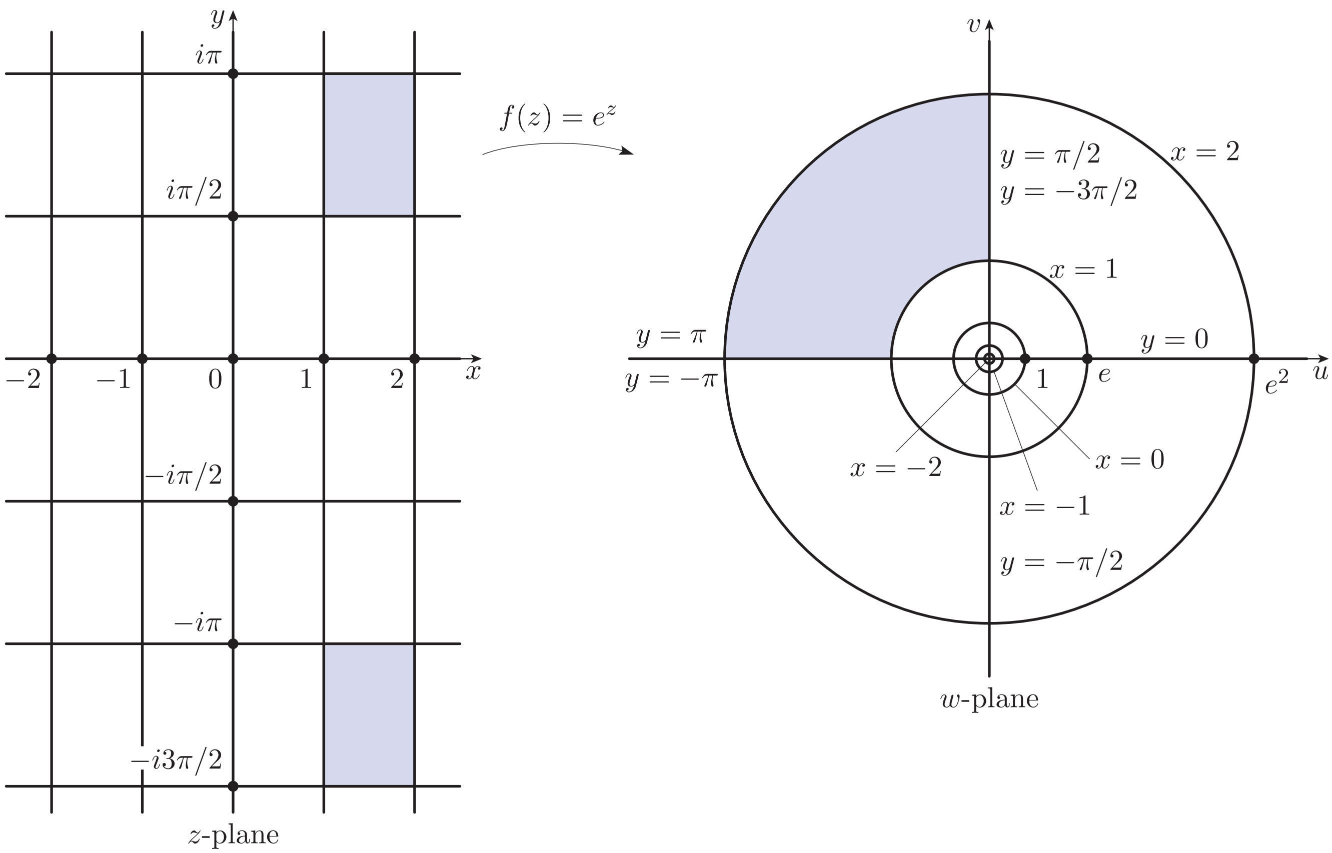


Figure 4.6 Image of a Cartesian grid under $f(z) = e^z$

Finally, notice that Figure 4.6 reveals an important property of exp, illustrated in Figure 4.7, which will prove useful in Section 5, where we discuss possible inverse functions for the exponential function.

> The image of the strip $\{x + iy : -\pi < y \leq \pi\}$ under $f(z) = e^z$ is $\mathbb{C} - \{0\}$.

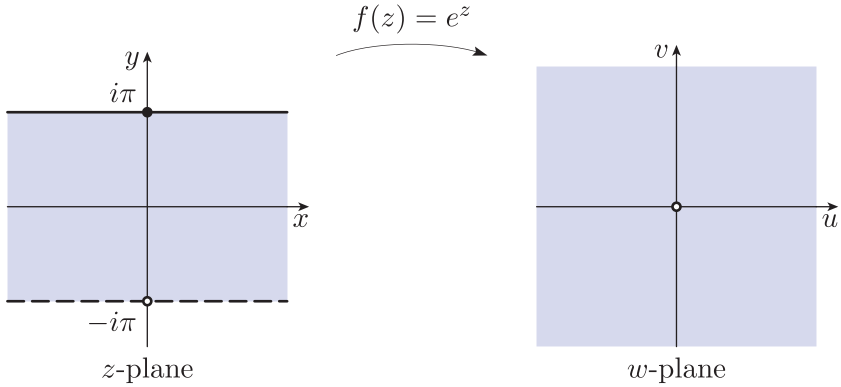


Figure 4.7 The image of the strip $\{x + iy : -\pi < y \leq \pi\}$ under $f(z) = e^z$ is $\mathbb{C} - \{0\}$

Exercise 4.4

Sketch the image of each of the following sets under the exponential function.

(a) $\{x + iy : -1 \leq x \leq 0,\ -\pi/4 \leq y \leq \pi/4\}$

(b) $\{x + iy : -1 \leq x \leq 1,\ \pi \leq y \leq 2\pi\}$

(c) $\{x + iy : 0 < y < 2\pi\}$

4.2 Trigonometric functions

In the study of real functions there seems, at first sight, to be no connection between the trigonometric functions $x \longmapsto \sin x$ and $x \longmapsto \cos x$, defined geometrically, and the exponential function $x \longmapsto e^x$. Certainly, their graphs (Figure 4.8) do not suggest any connection.

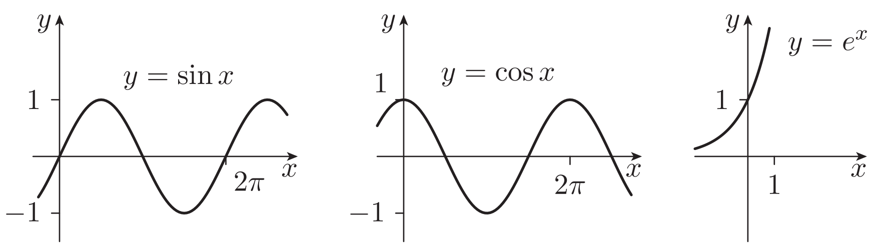


Figure 4.8 Graphs of $y = \sin x$, $y = \cos x$ and $y = e^x$

However, the definition of the *complex* exponential function makes use of real trigonometric functions, and it turns out that this complex exponential function can be used to define *complex* trigonometric functions.

The key to such a definition is Euler's Identity

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad \theta \in \mathbb{R},$$

together with the consequence

$$e^{-i\theta} = \cos\theta - i\sin\theta, \quad \theta \in \mathbb{R},$$

which holds because $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$. Eliminating first $\sin\theta$ and then $\cos\theta$ from these equations, we obtain

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \quad \text{and} \quad \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}). \tag{4.5}$$

These equations suggest the following definitions.

Definitions

For all z in $\mathbb{C}$,

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) \quad \text{and} \quad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}).$$

These functions are called the **cosine function** and **sine function**, respectively.

Remarks

1. We occasionally use the phrase '*complex* cosine function', rather than just 'cosine function', to emphasise that we are working with a complex function. (A similar comment applies to the sine function.)
2. If z is real, then these definitions give equations (4.5).

With these definitions, the functions cos and sin enjoy most (but not all) of the properties of the corresponding real trigonometric functions. Before stating these properties, we determine some values of these functions.

Example 4.2

Evaluate each of the following numbers in Cartesian form.

(a) $\sin i$ (b) $\cos(\pi + i)$

Solution

(a) $$\sin i = \frac{1}{2i}(e^{i^2} - e^{-i^2}) = \frac{1}{2i}(e^{-1} - e^{1}) = \frac{1}{2}(e - e^{-1})i$$

(b) $$\begin{aligned}\cos(\pi + i) &= \tfrac{1}{2}(e^{i(\pi+i)} + e^{-i(\pi+i)}) \\ &= \tfrac{1}{2}(e^{i\pi-1} + e^{-i\pi+1}) \\ &= \tfrac{1}{2}(e^{i\pi}e^{-1} + e^{-i\pi}e^{1}) \\ &= \tfrac{1}{2}(-e^{-1} - e^{1}) = -\tfrac{1}{2}(e + e^{-1}),\end{aligned}$$

using the fact that $e^{i\pi} = e^{-i\pi} = -1$.

Remarks

1. Notice that

$$|\sin i| = \tfrac{1}{2}(e - e^{-1}) \approx 1.175 > 1$$

and

$$|\cos(\pi + i)| = \tfrac{1}{2}(e + e^{-1}) \approx 1.543 > 1.$$

Thus the well-known properties of the real sine and cosine functions

$$|\sin x| \le 1 \quad \text{and} \quad |\cos x| \le 1, \quad \text{where } x \in \mathbb{R},$$

do not always hold when x is replaced by a complex number z.

2. The solutions to Example 4.2 suggest that there is a connection between the complex trigonometric functions and the hyperbolic functions. This will be made clear later in the section.

Exercise 4.5

Evaluate each of the following complex numbers in Cartesian form.

(a) $\sin(\pi/2 + i)$ (b) $\cos i$

In order to describe the algebraic properties of the complex sine and cosine functions, we need to introduce the full range of complex trigonometric functions. First, however, we determine the zeros of sin and cos.

Theorem 4.2

(a) The set of zeros of the sine function is

$$\{z : \sin z = 0\} = \{n\pi : n \in \mathbb{Z}\}.$$

(b) The set of zeros of the cosine function is

$$\{z : \cos z = 0\} = \{(n + \tfrac{1}{2})\pi : n \in \mathbb{Z}\}.$$

The theorem says that the only zeros of the complex sine and cosine functions are those of the real sine and cosine functions.

Proof

(a) Using the definition of $\sin z$, we have

$$\begin{aligned}
\sin z = 0 &\iff \frac{1}{2i}(e^{iz} - e^{-iz}) = 0 \\
&\iff e^{iz} = e^{-iz} \\
&\iff e^{2iz} = 1 \\
&\iff 2iz \in \{2n\pi i : n \in \mathbb{Z}\} \qquad \text{(see Exercise 4.2(d))} \\
&\iff z \in \{n\pi : n \in \mathbb{Z}\},
\end{aligned}$$

which proves part (a).

(b) Using the definition of $\cos z$, we have

$$\begin{aligned}\cos z = 0 &\iff \frac{1}{2}(e^{iz} + e^{-iz}) = 0\\ &\iff e^{iz} = -e^{-iz}\\ &\iff e^{2iz} = -1\\ &\iff 2iz \in \{(2n+1)\pi i : n \in \mathbb{Z}\} \quad \text{(see Exercise 4.2(d))}\\ &\iff z \in \{(n+\tfrac{1}{2})\pi : n \in \mathbb{Z}\},\end{aligned}$$

which proves part (b). ■

The other complex trigonometric functions tan, sec, cot and cosec are defined as in the real case.

Definitions

For all z in $\mathbb{C} - \{(n+\frac{1}{2})\pi : n \in \mathbb{Z}\}$,

$$\tan z = \frac{\sin z}{\cos z} \quad \text{and} \quad \sec z = \frac{1}{\cos z}.$$

For all z in $\mathbb{C} - \{n\pi : n \in \mathbb{Z}\}$,

$$\cot z = \frac{\cos z}{\sin z} \quad \text{and} \quad \operatorname{cosec} z = \frac{1}{\sin z}.$$

We now record the basic algebraic identities satisfied by these complex trigonometric functions. All of these identities are the same as identities satisfied by the real trigonometric functions.

Theorem 4.3 Trigonometric Identities

(a) **Addition**

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$$
$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$
$$\tan(z_1 + z_2) = \frac{\tan z_1 + \tan z_2}{1 - \tan z_1 \tan z_2}$$

(b) **Squares**

$$\cos^2 z + \sin^2 z = 1$$
$$\sec^2 z = 1 + \tan^2 z$$
$$\operatorname{cosec}^2 z = 1 + \cot^2 z$$

(c) **Negatives**

$$\sin(-z) = -\sin z$$
$$\cos(-z) = \cos z$$
$$\tan(-z) = -\tan z$$

(d) **Periodicity**

$\sin(z + 2\pi) = \sin z$

$\cos(z + 2\pi) = \cos z$

$\tan(z + \pi) = \tan z$

We prove three of these identities in the next example, and ask you to check some more of them in Exercise 4.6.

Example 4.3

Prove the following identities.

(a) $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$

(b) $\cos(-z) = \cos z$

(c) $\sin(z + 2\pi) = \sin z$

Solution

(a) Starting with the right-hand side, we have

$$\begin{aligned}
&\sin z_1 \cos z_2 + \cos z_1 \sin z_2 \\
&= \frac{1}{2i}(e^{iz_1} - e^{-iz_1})\frac{1}{2}(e^{iz_2} + e^{-iz_2}) + \frac{1}{2}(e^{iz_1} + e^{-iz_1})\frac{1}{2i}(e^{iz_2} - e^{-iz_2}) \\
&= \frac{1}{4i}\Big((e^{i(z_1+z_2)} + e^{i(z_1-z_2)} - e^{-i(z_1-z_2)} - e^{-i(z_1+z_2)}) \\
&\qquad + (e^{i(z_1+z_2)} - e^{i(z_1-z_2)} + e^{-i(z_1-z_2)} - e^{-i(z_1+z_2)})\Big) \\
&= \frac{1}{2i}(e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}) \\
&= \sin(z_1 + z_2),
\end{aligned}$$

as required.

(b) We have

$$\cos(-z) = \tfrac{1}{2}(e^{i(-z)} + e^{-i(-z)}) = \tfrac{1}{2}(e^{iz} + e^{-iz}) = \cos z,$$

as required.

(c) We have

$$\begin{aligned}
\sin(z + 2\pi) &= \frac{1}{2i}(e^{i(z+2\pi)} - e^{-i(z+2\pi)}) \\
&= \frac{1}{2i}(e^{iz}e^{2\pi i} - e^{-iz}e^{-2\pi i}) \\
&= \frac{1}{2i}(e^{iz} - e^{-iz}) \quad (\text{since } e^{2\pi i} = e^{-2\pi i} = 1) \\
&= \sin z,
\end{aligned}$$

as required.

Theorem 4.3 is by no means an exhaustive list of trigonometric identities. For example, we have not included identities such as

$$\sin(z_1 - z_2) = \sin z_1 \cos z_2 - \cos z_1 \sin z_2,$$
$$\cos(z_1 - z_2) = \cos z_1 \cos z_2 + \sin z_1 \sin z_2$$

and

$$\sin 2z = 2 \sin z \cos z.$$

However, these can readily be deduced from the identities in Theorem 4.3.

Exercise 4.6

(a) Prove the following identities.

(i) $\sin(-z) = -\sin z$ (ii) $\cos(z + 2\pi) = \cos z$

(b) Deduce the following identities from Theorem 4.3.

(i) $\cos 2z = 2\cos^2 z - 1$ (ii) $\tan(z_1 - z_2) = \dfrac{\tan z_1 - \tan z_2}{1 + \tan z_1 \tan z_2}$

4.3 Hyperbolic functions

Earlier in this section we referred to a relationship between complex trigonometric functions and the real hyperbolic functions

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}) \quad \text{and} \quad \cosh x = \tfrac{1}{2}(e^x + e^{-x}),$$

whose graphs appear in Figure 4.9.

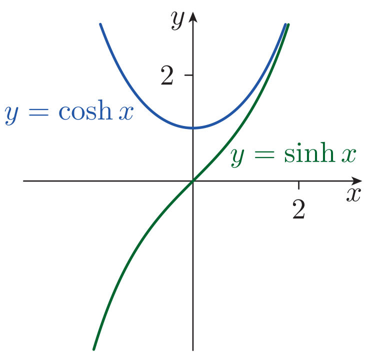


Figure 4.9 Graphs of $y = \sinh x$ and $y = \cosh x$

The complex hyperbolic functions are defined by the same formulas as for the real hyperbolic functions.

Definitions

For all z in $\mathbb{C}$,

$$\sinh z = \frac{1}{2}(e^z - e^{-z}) \quad \text{and} \quad \cosh z = \frac{1}{2}(e^z + e^{-z}).$$

For all z in $\mathbb{C} - \{(n + \frac{1}{2})\pi i : n \in \mathbb{Z}\}$,

$$\tanh z = \frac{\sinh z}{\cosh z} \quad \text{and} \quad \operatorname{sech} z = \frac{1}{\cosh z}.$$

For all z in $\mathbb{C} - \{n\pi i : n \in \mathbb{Z}\}$,

$$\coth z = \frac{\cosh z}{\sinh z} \quad \text{and} \quad \operatorname{cosech} z = \frac{1}{\sinh z}.$$

In these definitions we have used the facts that

$$\{z : \sinh z = 0\} = \{n\pi i : n \in \mathbb{Z}\}$$

and

$$\{z : \cosh z = 0\} = \{(n + \tfrac{1}{2})\pi i : n \in \mathbb{Z}\}.$$

All these zeros lie on the imaginary axis.

These 'zero sets' are readily deduced from the zero sets of the sine and cosine functions by using the following result, which shows the close relationship between the complex hyperbolic functions and the complex trigonometric functions.

Theorem 4.4

For all z in $\mathbb{C}$,

$$\sin(iz) = i\sinh z \quad \text{and} \quad \cos(iz) = \cosh z.$$

Proof For $z \in \mathbb{C}$,

$$\sin(iz) = \frac{1}{2i}(e^{i(iz)} - e^{-i(iz)}) = -\frac{1}{2}i(e^{-z} - e^{z}) = i\sinh z$$

and

$$\cos(iz) = \frac{1}{2}(e^{i(iz)} + e^{-i(iz)}) = \frac{1}{2}(e^{-z} + e^{z}) = \cosh z.$$

■

The hyperbolic functions satisfy a number of basic identities, summarised in Theorem 4.5. We omit the proofs, which can all be deduced either from Theorem 4.3, by using the identities in Theorem 4.4, or directly from the definitions.

Theorem 4.5 Hyperbolic Identities

(a) **Addition**

$$\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$$

$$\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$$

$$\tanh(z_1 + z_2) = \frac{\tanh z_1 + \tanh z_2}{1 + \tanh z_1 \tanh z_2}$$

(b) **Squares**

$$\cosh^2 z - \sinh^2 z = 1$$

$$\operatorname{sech}^2 z = 1 - \tanh^2 z$$

$$\operatorname{cosech}^2 z = \coth^2 z - 1$$

(c) **Negatives**

$$\sinh(-z) = -\sinh z$$

$$\cosh(-z) = \cosh z$$

$$\tanh(-z) = -\tanh z$$

(d) **Periodicity**

$$\sinh(z + 2\pi i) = \sinh z$$

$$\cosh(z + 2\pi i) = \cosh z$$

$$\tanh(z + \pi i) = \tanh z$$

The following example and exercise show that the hyperbolic functions play an important role in the determination of the real and imaginary parts of $\sin z$ and $\cos z$.

Example 4.4

Let $z = x + iy$. Prove the following identities.

(a) $\sin z = \sin x \cosh y + i \cos x \sinh y$ (b) $|\sin z|^2 = \sin^2 x + \sinh^2 y$

Solution

(a) We have

$$\begin{aligned}
&\sin(x + iy) \\
&= \sin x \cos(iy) + \cos x \sin(iy) && \text{(Theorem 4.3(a))} \\
&= \sin x \cosh y + i \cos x \sinh y && \text{(Theorem 4.4).}
\end{aligned}$$

(b) We have

$$\begin{aligned}
&|\sin(x + iy)|^2 \\
&= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y && \text{(part (a))} \\
&= \sin^2 x(1 + \sinh^2 y) + \cos^2 x \sinh^2 y && \text{(Theorem 4.5(b))} \\
&= \sin^2 x + \sinh^2 y\,(\sin^2 x + \cos^2 x) \\
&= \sin^2 x + \sinh^2 y.
\end{aligned}$$

Exercise 4.7

Let $z = x + iy$. Prove the following identities.

(a) $\cos z = \cos x \cosh y - i \sin x \sinh y$ (b) $|\cos z|^2 = \cos^2 x + \sinh^2 y$

Further exercises

Exercise 4.8

Express the following complex numbers in Cartesian form.

(a) $e^{3\pi i}$ (b) $ee^{\pi i/2}$ (c) $e^{2\pi i/3}$ (d) $e^{-3\pi i/2}$ (e) $e^{2+\pi i}$

(f) $e^{3+\pi i/2}$ (g) $e^{(\pi i/6)-1}$ (h) $e^{(\cos\theta + i \sin\theta)}$

Exercise 4.9

(a) Express the following complex numbers in the polar form $re^{i\theta}$.

(i) $\dfrac{1}{\sqrt{2}} - \dfrac{i}{\sqrt{2}}$ (ii) $-1 - i$ (iii) $1 + \sqrt{3}i$

(b) Hence evaluate the following complex numbers, giving your answers in Cartesian form.

(i) $\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)^3$ (ii) $(1+\sqrt{3}i)^{-7}$

Exercise 4.10

Express the following complex numbers in Cartesian form.

(a) $\sin(\pi + 2i)$ (b) $\cos(\pi/2 - i)$ (c) $\tan i$

In parts (a) and (b) you can work from the definitions of sin and cos, or use identities established in the section.

Exercise 4.11

Prove the following identities.

(a) $\overline{e^z} = e^{\overline{z}}$ (b) $\sin 2z = 2\sin z \cos z$ (c) $\overline{\sin z} = \sin \overline{z}$

(d) $\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$

(e) $\cosh^2 z - \sinh^2 z = 1$

In parts (a), (b) and (c), work from the definitions of e^z, $\sin 2z$ and $\sin z$; in parts (d) and (e), use identities established in this section.

5 Logarithms and powers

After working through this section, you should be able to:

- determine the *principal logarithm* $\operatorname{Log} z$ of a non-zero complex number z, and describe the geometric effect of the function $z \longmapsto \operatorname{Log} z$
- determine the *principal power* z^α, where $\alpha \in \mathbb{C}$, of a non-zero complex number z.

5.1 Logarithms of complex numbers

In real analysis the natural logarithm function log (that is, the logarithm to base e, sometimes denoted ln or $\log_e$) is defined as the inverse function of the exponential function. Since $x \longmapsto e^x$ is a one-to-one function on $\mathbb{R}$ with image set $(0,\infty)$, the inverse function log has domain $(0,\infty)$, and is defined by the rule

$$\log y = x, \quad \text{where } y = e^x.$$

The graph of the function log can be obtained by reflecting the graph of $y = e^x$ in the line $y = x$ (Figure 5.1).

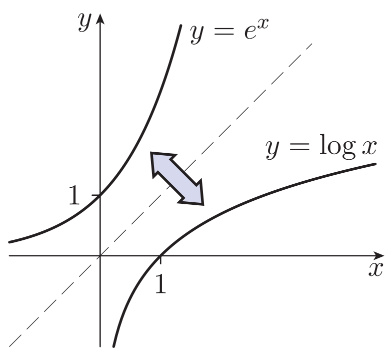


Figure 5.1 Graphs of $y = e^x$ and $y = \log x$

Now consider the complex exponential function $f(z) = e^z$. In trying to define an inverse function f^{-1} for f, we encounter the fact that f is not a one-to-one function. For example,

$$\cdots = f(-2\pi i) = f(0) = f(2\pi i) = f(4\pi i) = \cdots = 1.$$

To get around this difficulty, we seek a set A (preferably as large as possible) on which $f(z) = e^z$ *is* one-to-one. We then restrict f to the set A, retaining the name f for the restricted function with domain A, and define an inverse function f^{-1} with domain $f(A)$. This approach was used in Subsection 1.5 with the function $f(z) = z^2$. We will find (just as we did for the function $f(z) = z^2$) that there are many choices for the set A. The choice in the following example is motivated by the periodicity property

$$e^{z+2\pi i} = e^z, \quad \text{for all } z \text{ in } \mathbb{C},$$

which suggests that A should be chosen in such a way that it does not contain two points that differ by $2\pi i$.

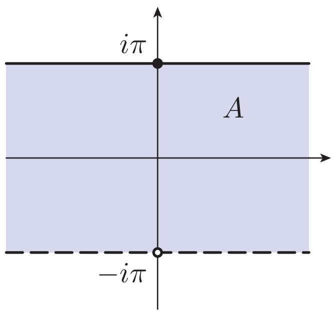


Figure 5.2 A horizontal strip

Example 5.1

Let

$$A = \{x + iy : -\pi < y \leq \pi\},$$

which is a horizontal strip (Figure 5.2). Prove that the function

$$f(z) = e^z \quad (z \in A)$$

has an inverse function f^{-1}, and determine the domain and rule of f^{-1}.

Solution

First we determine the image set of f:

$$\begin{aligned} f(A) &= \{e^z : z \in A\} \\ &= \{w = e^{x+iy} : x \in \mathbb{R},\ -\pi < y \leq \pi\} \\ &= \{w = e^x e^{iy} : x \in \mathbb{R},\ -\pi < y \leq \pi\} \\ &= \{w = \rho e^{i\phi} : \rho > 0,\ -\pi < \phi \leq \pi\} \\ &= \mathbb{C} - \{0\}, \end{aligned}$$

where $\rho = e^x$ and $\phi = y$. (In fact, we have already observed that $f(A) = \mathbb{C} - \{0\}$ in Figure 4.7.)

Now, for each $w \in \mathbb{C} - \{0\}$, we wish to solve the equation

$$w = e^z \tag{5.1}$$

to obtain a unique solution z in A.

Each w in $\mathbb{C} - \{0\}$ can be written in the form

$$w = \rho e^{i\phi}, \quad \text{where } \rho > 0 \text{ and } -\pi < \phi \leq \pi,$$

and equation (5.1) is then

$$\rho e^{i\phi} = e^z = e^x e^{iy}, \quad \text{where } z = x + iy.$$

Equating moduli on both sides of the equation $\rho e^{i\phi} = e^x e^{iy}$, we obtain $\rho = e^x$, and on dividing both sides by ρ we obtain $e^{i\phi} = e^{iy}$.

Hence

$$x = \log \rho \quad \text{and} \quad y = \phi + 2n\pi,$$

where $n \in \mathbb{Z}$. For $n = 0$, the solution is

$$z = x + iy = \log \rho + i\phi,$$

which lies in A, since $-\pi < \phi \leq \pi$, whereas the other solutions (with $n \neq 0$) lie outside A.

Thus f is a one-to-one function, with image set $\mathbb{C} - \{0\}$. Hence f has an inverse function f^{-1} with domain $\mathbb{C} - \{0\}$ and rule

$$f^{-1}(w) = \log \rho + i\phi,$$

where $w = \rho e^{i\phi}$ and $\rho > 0$, $-\pi < \phi \leq \pi$.

Remarks

1. Since $\phi = \operatorname{Arg} w$ and $\rho = |w|$ in this solution, the rule for f^{-1} can also be written in the form

$$f^{-1}(w) = \log |w| + i \operatorname{Arg} w \quad (w \neq 0).$$

2. Example 5.1 is important because it will be used shortly in defining logarithms of complex numbers.

Exercise 5.1

Let $A = \{x + iy : 0 \leq y < 2\pi\}$. Prove that the function

$$f(z) = e^z \quad (z \in A)$$

has an inverse function f^{-1}, and determine the domain and rule of f^{-1}.

The solution to Example 5.1 and Remark 1 that follows it show that if $w \neq 0$, then the equation $e^z = w$ has infinitely many solutions of the form

$$z = \log |w| + i(\operatorname{Arg} w + 2n\pi), \quad n \in \mathbb{Z}.$$

Each of these solutions is called a **logarithm** of w. The infinitely many logarithms of w correspond to the infinitely many arguments of w, which are all of the form

$$\operatorname{Arg} w + 2n\pi, \quad n \in \mathbb{Z}.$$

Some texts use the notation '$\log w$' to denote a particular logarithm of w, determined by a particular choice of argument. However, to prevent confusion, we will avoid this ambiguous notation, and instead we will almost always use the logarithm of w that corresponds to the principal argument of w (that is, $n = 0$). This solution,

$$z = \log |w| + i \operatorname{Arg} w,$$

of $e^z = w$ is called the *principal logarithm* of w, written $\operatorname{Log} w$. (Note the capital L in Log, corresponding to the capital A in Arg.)

Thus the inverse function f^{-1} of Example 5.1 can be written as

$$f^{-1}(w) = \operatorname{Log} w \quad (w \in \mathbb{C} - \{0\}).$$

Definitions

For $z \in \mathbb{C} - \{0\}$, the **principal logarithm** of z is

$$\operatorname{Log} z = \log|z| + i \operatorname{Arg} z.$$

The corresponding **principal logarithm function** is called Log.

Remarks

1. If z is real and positive (that is, $z = x + 0i$, where $x > 0$), then

   $$\operatorname{Log} z = \operatorname{Log} x = \log x,$$

   where $\log x$ denotes the usual real logarithm of x, as expected. Thus the restriction of Log to $(0, \infty)$ is log.
2. Note that the definition of Log applies if z is a negative real number. For example,

   $$\operatorname{Log}(-2) = \log|-2| + i \operatorname{Arg}(-2) = \log 2 + i\pi.$$
3. Since Log is the inverse function of the function

   $$f(z) = e^z \quad (z \in \{x + iy : -\pi < y \le \pi\}),$$

   we have the following two identities.

   $$e^{\operatorname{Log} z} = z, \quad \text{for } z \in \mathbb{C} - \{0\},$$
   $$\operatorname{Log}(e^z) = z, \quad \text{for } z \in \{x + iy : -\pi < y \le \pi\}.$$

   The latter identity is false if z lies outside the strip

   $$\{x + iy : -\pi < y \le \pi\}.$$

   For example, if $z = 2\pi i$, then

   $$\operatorname{Log}(e^{2\pi i}) = \operatorname{Log} 1 = \log 1 = 0 \ne 2\pi i.$$
4. The function Log has domain $\mathbb{C} - \{0\}$, and its image set, written in terms of w, is $\{w : -\pi < \operatorname{Im} w \le \pi\}$, as shown in Figure 5.3.

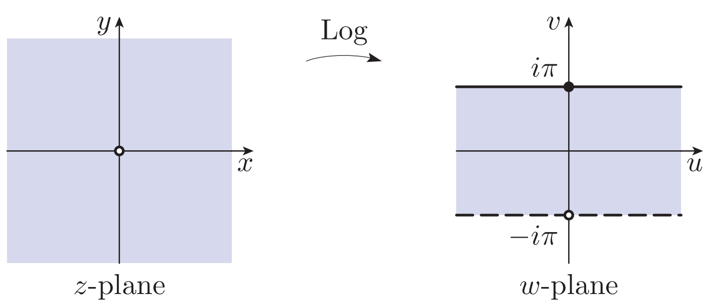


Figure 5.3 Log has domain $\mathbb{C} - \{0\}$ and image set $\{x + iy : -\pi < y \le \pi\}$

Example 5.2

Evaluate $\mathrm{Log}(-1+i)$ in Cartesian form.

Solution

Since $|-1+i| = \sqrt{2}$ and $\mathrm{Arg}(-1+i) = 3\pi/4$, we have

$$\mathrm{Log}(-1+i) = \log\sqrt{2} + i3\pi/4.$$

Exercise 5.2

Evaluate the following complex numbers in Cartesian form.

(a) $\mathrm{Log}\, i$ (b) $\mathrm{Log}(\sqrt{3}-i)$ (c) $\mathrm{Log}(\frac{1}{2}+\frac{1}{2}i)$

The real function log satisfies various identities, such as

$$\log(x_1x_2) = \log x_1 + \log x_2, \quad \text{for } x_1, x_2 > 0,$$

and

$$\log(1/x) = -\log x, \quad \text{for } x > 0.$$

It is natural to hope that similar identities will hold for the complex function Log, and this is indeed the case, provided that suitable restrictions are placed on the variables involved.

Theorem 5.1 Logarithmic Identities

(a) **Multiplication**

$$\mathrm{Log}(z_1z_2) = \mathrm{Log}\, z_1 + \mathrm{Log}\, z_2, \quad \text{if } \mathrm{Arg}\, z_1, \mathrm{Arg}\, z_2 \in (-\pi/2, \pi/2].$$

(b) **Reciprocals**

$$\mathrm{Log}(1/z) = -\mathrm{Log}\, z, \quad \text{if } \mathrm{Arg}\, z \in (-\pi, \pi).$$

Part (b) does not hold if $\mathrm{Arg}\, z = \pi$, as you can check by choosing $z = -1$.

Proof

(a) If $\mathrm{Arg}\, z_1, \mathrm{Arg}\, z_2 \in (-\pi/2, \pi/2]$, then $\mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2 \in (-\pi, \pi]$, so

$$\mathrm{Arg}(z_1z_2) = \mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2.$$

(You met this property of Arg in Subsection 2.3 of Unit A1.) Hence

$$\begin{aligned}\mathrm{Log}(z_1z_2) &= \log|z_1z_2| + i\,\mathrm{Arg}(z_1z_2)\\ &= \log|z_1| + \log|z_2| + i\,(\mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2)\\ &= (\log|z_1| + i\,\mathrm{Arg}\, z_1) + (\log|z_2| + i\,\mathrm{Arg}\, z_2)\\ &= \mathrm{Log}\, z_1 + \mathrm{Log}\, z_2.\end{aligned}$$

(b) Since $1/z = \overline{z}/|z|^2$ and $\operatorname{Arg} z \neq \pi$, it follows that

$$\operatorname{Arg}(1/z) = \operatorname{Arg} \overline{z} = -\operatorname{Arg} z.$$

Hence

$$\begin{aligned}\operatorname{Log}(1/z) &= \log|1/z| + i \operatorname{Arg}(1/z)\\ &= \log(1/|z|) + i \operatorname{Arg}(1/z)\\ &= -\log|z| - i \operatorname{Arg} z\\ &= -\operatorname{Log} z.\end{aligned}$$

■

Remark

Using properties of Arg described in Subsection 2.3 of Unit A1, it can be shown that the identity in Theorem 5.1(a) holds in the following form for any values in the domain $\mathbb{C} - \{0\}$ of Log:

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2 + 2n\pi i, \tag{5.2}$$

where n is -1, 0 or 1, depending on whether $\operatorname{Arg} z_1 + \operatorname{Arg} z_2$ is greater than π, lies in the interval $(-\pi, \pi]$, or is less than or equal to $-\pi$.

For example, if $z_1 = z_2 = -1$, then

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log} 1 = 0,$$

$$\operatorname{Log} z_1 + \operatorname{Log} z_2 = (\log 1 + i\pi) + (\log 1 + i\pi) = 2\pi i.$$

Thus the identity (5.2) holds with $n = -1$ in this case.

The geometric effect of the function Log

We now briefly discuss the geometric effect of the function Log, drawing on our knowledge of the geometric effect of the exponential function $z \longmapsto e^z$ obtained in Subsection 4.1 and depicted in Figure 4.6. Since Log is the inverse function of the restriction of the exponential function to the horizontal strip $\{z : -\pi < \operatorname{Im} z \leq \pi\}$, we can understand the geometric effect of Log by 'reversing' the effect of exp observed in Figure 4.6. Thus Log maps circles with centre 0 and radius r onto line segments of the form $u = \log r$, $-\pi < v \leq \pi$ (where $w = u + iv$), and it maps rays $\operatorname{Arg} z = \theta$ onto lines of the form $v = \theta$. Figure 5.4 shows the image of this polar grid under the function Log.

Notice that

- points lying outside the unit circle $\{z : |z| = 1\}$ have images lying in the right half-strip, whereas non-zero points inside the unit circle have images lying in the left half-strip
- the Log function behaves in a rather strange manner near the negative real axis. For example, as z approaches the point -1 on the negative real axis from *above*, the image point $w = \operatorname{Log} z$ approaches the point $i\pi$, but if z approaches -1 from *below*, then $w = \operatorname{Log} z$ approaches the point $-i\pi$ (which is not in the image set of Log). This strange behaviour occurs near the negative real axis because of the particular definition of Arg that we have chosen (and the fact that $\operatorname{Log} z = \log|z| + i \operatorname{Arg} z$). We consider this behaviour further in Unit A3.

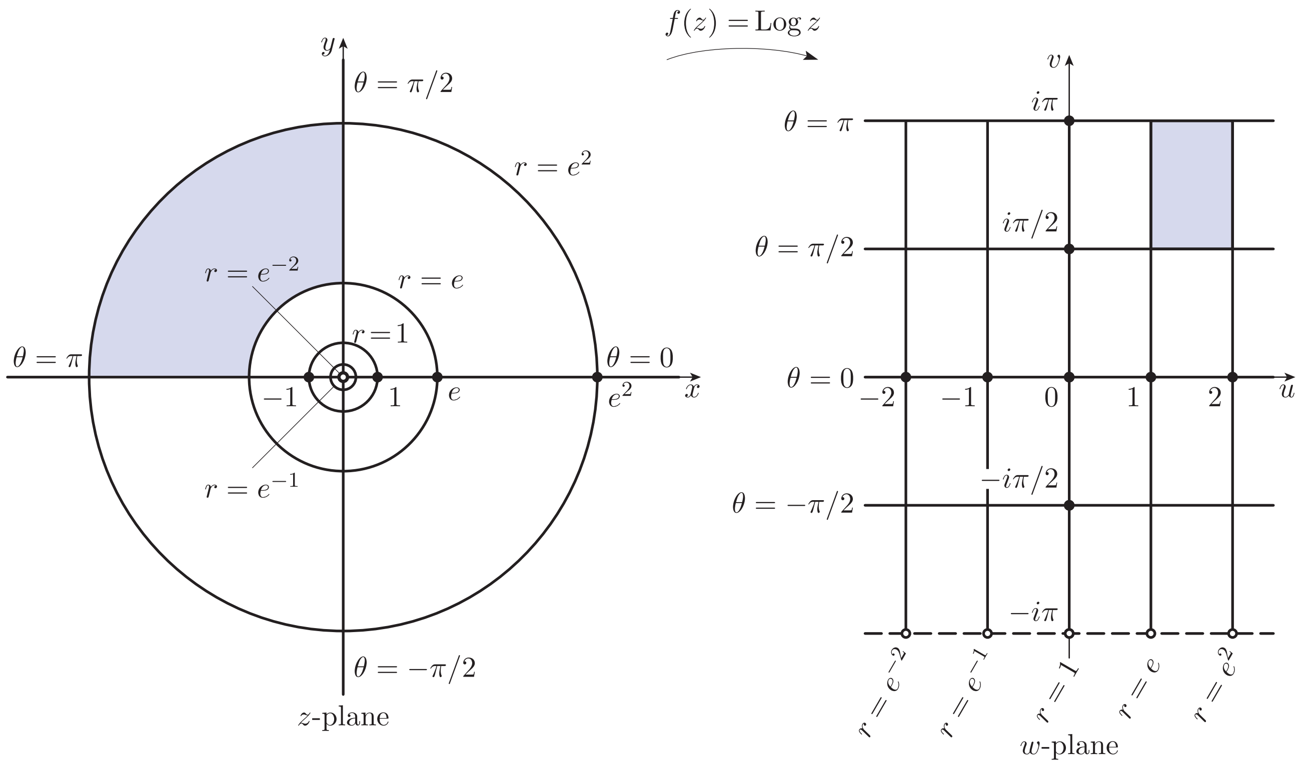


Figure 5.4 Image of a polar grid under $f(z) = \operatorname{Log} z$

Exercise 5.3

Classify the following statements as True or False.

(a) The image under the function Log of the ellipse

$$4x^2 + 9y^2 = 1$$

lies in the right half-plane.

(b) The image under the function Log of the ray $\theta = \pi/4$ lies in the right half-plane.

(c) There is a point $z \in \mathbb{C}$ such that

$$\operatorname{Log} z = 1 + 4i.$$

(d) There is a point $z \in \mathbb{C}$ such that

$$\operatorname{Log} z = 1 + \tfrac{1}{4}i.$$

Leonhard Euler

Leonhard Euler and logarithms

The first person to recognise the many-valued nature of complex logarithms was the prolific Swiss mathematician and scientist Leonhard Euler (1707–1783). In a letter to another Swiss mathematician Gabriel Cramer (1704–1752) in 1746, Euler wrote:

> I have finally discovered the true solution: in the same way that to one sine there correspond an infinite number of different angles I have found that it is the same with logarithms, and each number has an infinity of different logarithms, all of them imaginary unless the number is real and positive; there is only one logarithm that is real, and we regard it as its unique logarithm.
>
> (Speziali, 1983, p. 428, cited in Bottazzini and Gray, 2013, p. 82)

In Euler's time, 'imaginary' meant 'complex and not real'.

As the quote demonstrates, Euler's understanding of logarithms was remarkably advanced; his work on complex numbers led to considerable developments in the subject. He was the first to publish the equation $e^{ix} = \cos x + i \sin x$ (Euler's Identity) which he obtained not by defining e^{ix} from that formula as we have done, but by defining the exponential and trigonometric functions in terms of series, and then proceeding with a method similar to that presented at the start of Subsection 4.1.

5.2 Powers of complex numbers

In this subsection we define the expression z^α, where z is any *non-zero* complex number and α is any complex number. In Subsection 3.1 of Unit A1 you saw that any non-zero complex number z has n nth roots and that the expression $z^{1/n}$ is reserved for just one of these roots, called the *principal nth root of z*. It seems likely, therefore, that there is going to be some difficulty in defining the expression z^α in a unique way.

Recall first that if $a > 0$ and $x \in \mathbb{R}$, then a^x satisfies the equation

$$a^x = e^{x \log a}.$$

It is tempting to define z^α by means of a similar formula, namely

$$\text{‘}z^\alpha = e^{\alpha \log z}\text{’}, \quad \text{where ‘}\log z\text{’ is a logarithm of } z.$$

The problem is, however, that any non-zero complex number z has infinitely many logarithms, so the formula above would give rise to infinitely many possible values of z^α. To avoid confusion, we will define z^α using $\operatorname{Log} z$, the principal logarithm of z.

Definitions

For $z, \alpha \in \mathbb{C}$, with $z \neq 0$, the **principal αth power** of z is

$$z^\alpha = \exp(\alpha \operatorname{Log} z).$$

The function $z \longmapsto z^\alpha$ is called the **principal αth power function**.

Remarks

1. It can be shown that this definition agrees with the usual meaning of z^α if $\alpha = n$ or $\alpha = 1/n$, where n is a positive integer. For example, if $\alpha = n$ and $z \neq 0$, then
$$\begin{aligned} e^{n \operatorname{Log} z} &= e^{\operatorname{Log} z + \cdots + \operatorname{Log} z} \\ &= e^{\operatorname{Log} z} \times \cdots \times e^{\operatorname{Log} z} \\ &= z \times \cdots \times z = z^n. \end{aligned}$$
2. This definition assigns no value to 0^α. However, in Subsection 3.1 of Unit A1, we defined 0^n and $0^{1/n}$ to be 0, for $n = 1, 2, 3, \ldots$.
3. Some texts take a different approach and allow both '$\log z$' and z^α to represent infinitely many different values, and specify, when appropriate, which value is being considered at a given time.

Example 5.3

Express each of the following numbers in Cartesian form.

(a) $(-1)^{1/2}$ (b) $(1+i)^i$ (c) i^i

Solution

(a) We have
$$\begin{aligned} (-1)^{1/2} &= \exp\left(\tfrac{1}{2} \operatorname{Log}(-1)\right) \\ &= e^{i\pi/2} \qquad (\text{since } \operatorname{Log}(-1) = i\pi) \\ &= i, \end{aligned}$$
as expected!

(b) Since $1 + i = \sqrt{2}(\cos \pi/4 + i \sin \pi/4)$, we have
$$\begin{aligned} (1+i)^i &= \exp(i \operatorname{Log}(1+i)) \\ &= \exp(i(\log \sqrt{2} + i\pi/4)) \\ &= \exp(-\pi/4 + i \log \sqrt{2}) \\ &= e^{-\pi/4}(\cos(\log \sqrt{2}) + i \sin(\log \sqrt{2})). \end{aligned}$$

(c) We have
$$\begin{aligned} i^i &= \exp(i \operatorname{Log} i) \\ &= \exp(i(i\pi/2)) \quad (\text{since } \operatorname{Log} i = i\pi/2) \\ &= e^{-\pi/2}, \end{aligned}$$
a real number!

Exercise 5.4

Express each of the following numbers in Cartesian form.

(a) $(1+i)^{2/3}$ (b) i^{1+i}

Exercise 5.5

Show that for $\alpha = 1/n$, where n is a positive integer, the definition of z^α given above agrees with the definition of $z^{1/n}$ given in Subsection 3.1 of Unit A1 (where we used the notation $\rho = |z|$ and $\phi = \operatorname{Arg} z$).

Exercise 5.6

(a) Find non-zero complex numbers z_1, z_2 and α for which the equation $z_1^\alpha z_2^\alpha = (z_1 z_2)^\alpha$ does not hold.

(b) Prove that $z^\alpha z^\beta = z^{\alpha+\beta}$ for all $z \in \mathbb{C} - \{0\}$ and $\alpha, \beta \in \mathbb{C}$.

Further exercises

Exercise 5.7

Express each of the following complex numbers in Cartesian form.

(a) $\operatorname{Log}(-2)$ (b) $\operatorname{Log}(i^3)$ (c) $\operatorname{Log}(1+i)$ (d) $\operatorname{Log}\sqrt{3}$

(e) $\operatorname{Log}(i-\sqrt{3})$ (f) $\operatorname{Log}\left(\dfrac{1-i}{\sqrt{2}}\right)$

Exercise 5.8

Express each of the following complex numbers in Cartesian form.

(a) i^{-i} (b) $(-i)^i$ (c) $(1-i)^i$ (d) $(-1)^i$

Solutions to exercises

Solution to Exercise 1.1

(a) Domain $\mathbb{C}$, codomain $\mathbb{C}$.

(b) Domain $\mathbb{C}-\{-2\}$, codomain $\mathbb{C}$.

(c) Domain $\mathbb{C}-\{0\}$, codomain $\mathbb{C}$.

(d) Domain $\mathbb{C}-\{-i,i\}$, codomain $\mathbb{C}$.

Solution to Exercise 1.2

(a) The domain of f is $\mathbb{C}$. The image set of f is

$$\begin{aligned}f(\mathbb{C}) &= \{3iz : z\in\mathbb{C}\}\\ &= \left\{w : z=\frac{w}{3i}\in\mathbb{C}\right\}\\ &= \{w : w\in\mathbb{C}\}\\ &= \mathbb{C}.\end{aligned}$$

(b) The domain of f is $\mathbb{C}-\{-i\}$. The image set of f is

$$\begin{aligned}f(\mathbb{C}-\{-i\}) &= \left\{\frac{3z+1}{z+i} : z\in\mathbb{C}-\{-i\}\right\}\\ &= \left\{w : z=\frac{1-iw}{w-3}\neq -i\right\}\\ &= \{w : w\neq 3\}\\ &= \mathbb{C}-\{3\},\end{aligned}$$

where we have used the fact that the equation

$$\frac{1-iw}{w-3}=-i,$$

or, equivalently, $1-iw=-iw+3i$, has no solutions.

(c) The domain of f is $\mathbb{C}$. The image set of f is

$$\begin{aligned}f(\mathbb{C}) &= \{\operatorname{Im} z : z\in\mathbb{C}\}\\ &= \{y : y\in\mathbb{R}\} \quad (z=x+iy)\\ &= \mathbb{R}.\end{aligned}$$

Solution to Exercise 1.3

(a) The domain of f is $\mathbb{C}$. The image set is

$$f(\mathbb{C})=\{x\in\mathbb{R} : x\geq 0\}.$$

(b) The domain of f is $\mathbb{C}-\{0\}$. The image set is

$$f(\mathbb{C}-\{0\})=(-\pi,\pi].$$

Solution to Exercise 1.4

(a) $f+g$ has domain $\mathbb{C}-\{0,1\}$ and rule

$$\begin{aligned}(f+g)(z) &= f(z)+g(z)\\ &= \frac{1}{z}+\frac{z+3i}{z^2-z}\\ &= \frac{2z-1+3i}{z^2-z}.\end{aligned}$$

(b) fg has domain $\mathbb{C}-\{0,1\}$ and rule

$$\begin{aligned}(fg)(z) &= f(z)g(z)\\ &= \frac{1}{z}\times\frac{z+3i}{z^2-z}\\ &= \frac{z+3i}{z^3-z^2}.\end{aligned}$$

(c) f/g has domain $\mathbb{C}-\{0,1,-3i\}$ and rule

$$\begin{aligned}(f/g)(z) &= \frac{f(z)}{g(z)}\\ &= \frac{1}{z}\bigg/\left(\frac{z+3i}{z^2-z}\right)\\ &= \frac{z-1}{z+3i}.\end{aligned}$$

(Note that 0 and 1 are excluded from the domain of f/g even though $(z-1)/(z+3i)$ is defined at these points.)

Solution to Exercise 1.5

(a) The domain of $g\circ f$ is

$$\begin{aligned}&\text{domain of } f-\left\{z : \frac{1}{z}\notin\mathbb{C}-\{0,1\}\right\}\\ &= (\mathbb{C}-\{0\})-\left\{z : \frac{1}{z}\in\{0,1\}\right\}\\ &= (\mathbb{C}-\{0\})-\{1\}=\mathbb{C}-\{0,1\}.\end{aligned}$$

The rule of $g\circ f$ is

$$g(f(z))=\frac{(1/z)+3i}{(1/z)^2-(1/z)}=\frac{z+3iz^2}{1-z}.$$

(b) The domain of $f\circ g$ is

$$\begin{aligned}&\text{domain of } g-\left\{z : \frac{z+3i}{z^2-z}\notin\mathbb{C}-\{0\}\right\}\\ &= (\mathbb{C}-\{0,1\})-\left\{z : \frac{z+3i}{z^2-z}\in\{0\}\right\}\\ &= (\mathbb{C}-\{0,1\})-\{-3i\}=\mathbb{C}-\{0,1,-3i\}.\end{aligned}$$

The rule of $f \circ g$ is

$$f(g(z)) = 1\Big/\left(\frac{z+3i}{z^2-z}\right) = \frac{z^2-z}{z+3i}.$$

Solution to Exercise 1.6

First we determine the image set of f. This is $\mathbb{C} - \{3\}$, from Exercise 1.2(b).

Now, for each $w \in \mathbb{C} - \{3\}$ we wish to solve the equation

$$w = \frac{3z+1}{z+i}$$

to obtain a unique solution z in $\mathbb{C} - \{-i\}$. This is achieved by the rearrangement

$$z = \frac{1-iw}{w-3}.$$

Thus f is a one-to-one function, with image set $\mathbb{C} - \{3\}$. Hence f has an inverse function f^{-1} with domain $\mathbb{C} - \{3\}$ and rule

$$f^{-1}(w) = \frac{1-iw}{w-3}.$$

Solution to Exercise 1.7

Let us first determine the image set of f, which is

$$f(A) = \{w = z^2 : z \in A\} = \{0\} \cup \{w = z^2 : 0 \le \operatorname{Arg} z < \pi\}.$$

By writing $z = r(\cos\theta + i\sin\theta)$, we see that $f(A)$ is equal to the union of $\{0\}$ and

$$\{w = r^2(\cos 2\theta + i\sin 2\theta) : r > 0, 0 \le \theta < \pi\}.$$

Let $\rho = r^2$ and $\phi = 2\theta$; then $f(A)$ is the union of $\{0\}$ and

$$\{w = \rho(\cos\phi + i\sin\phi) : \rho > 0, 0 \le \phi < 2\pi\},$$

so $f(A) = \mathbb{C}$.

Now, for each $w \in \mathbb{C}$ we wish to solve the equation

$$w = z^2 \tag{S1}$$

to obtain a unique solution z in A. If $w = 0$, then equation (S1) has the unique solution $z = 0$. If $w \ne 0$, then w can be written in the form

$$w = \rho(\cos\phi + i\sin\phi),$$

where $\rho > 0$ and $0 \le \phi < 2\pi$, and equation (S1) then has exactly two solutions

$$z_0 = \rho^{1/2}(\cos(\phi/2) + i\sin(\phi/2)),$$
$$z_1 = \rho^{1/2}(\cos(\phi/2+\pi) + i\sin(\phi/2+\pi)),$$

by Theorem 3.1 of Unit A1. Clearly, $z_0 \in A$, since $0 \le \phi/2 < \pi$, whereas $z_1 \notin A$.

Thus f is a one-to-one function, with image set $\mathbb{C}$. Hence f has an inverse function f^{-1} with domain $\mathbb{C}$ and rule given by $f^{-1}(0) = 0$ and

$$f^{-1}(w) = \rho^{1/2}(\cos(\phi/2) + i\sin(\phi/2)),$$

where $w = \rho(\cos\phi + i\sin\phi)$, $\rho > 0$, $0 \le \phi < 2\pi$.

Remark: Using the definition of the principal square root $\sqrt{w}$ given in Subsection 3.1 of Unit A1, we see that

$$f^{-1}(w) = \sqrt{w}$$

for values of w for which $0 \le \phi \le \pi$. However, $f^{-1}(w)$ and $\sqrt{w}$ differ for $\pi < \phi < 2\pi$. For example, the number $w = -i$ has polar form

$$\cos(3\pi/2) + i\sin(3\pi/2),$$

and $0 \le 3\pi/2 < 2\pi$, so

$$f^{-1}(-i) = \cos(3\pi/4) + i\sin(3\pi/4) = \frac{-1+i}{\sqrt{2}}.$$

However, $-i$ also has polar form

$$\cos(-\pi/2) + i\sin(-\pi/2),$$

and $-\pi < -\pi/2 \le \pi$, so

$$\sqrt{-i} = \cos(-\pi/4) + i\sin(-\pi/4) = \frac{1-i}{\sqrt{2}}.$$

Solution to Exercise 1.8

(a) $\mathbb{C}$

(b) $\mathbb{C} - \{1\}$

(c) $\mathbb{C} - \{-i, i\}$

(d) $\mathbb{C} - \{z : \operatorname{Re} z = 0\} = \{z : \operatorname{Re} z \ne 0\}$

(e) $\mathbb{C} - \{z : |z| = 1\} = \{z : |z| \ne 1\}$

(f) $\mathbb{C} - \left\{\frac{1}{2}(1+\sqrt{3}i), -1, \frac{1}{2}(1-\sqrt{3}i)\right\}$

Solution to Exercise 1.9

The image set of each f is determined as follows.

(a) $\{2z+1 : z \in \mathbb{C}\}$
$= \{w : z = (w-1)/2 \in \mathbb{C}\}$
$= \{w : w \in \mathbb{C}\} = \mathbb{C}$

(b) $\{1/(z-1) : z \in \mathbb{C} - \{1\}\}$
$= \{w : z = (1+w)/w \neq 1\}$
$= \{w : w \neq 0\} = \mathbb{C} - \{0\}$

(c) $\{z/(z-1) : z \in \mathbb{C} - \{1\}\}$
$= \{w : z = w/(w-1) \neq 1\}$
$= \{w : w \neq 1\} = \mathbb{C} - \{1\}$

(d) $\{|z-1| : z \in \mathbb{C}\} = \{r : r \geq 0\} = [0, \infty)$

(e) Writing $z = x + iy$, we see that

$$\{\operatorname{Re}(z+i) : z \in \mathbb{C}\} = \{x : x \in \mathbb{R}\} = \mathbb{R}.$$

(f) $\{|\operatorname{Arg} z| : z \in \mathbb{C} - \{0\}\}$
$= \{|\theta| : \theta \in (-\pi, \pi]\} = [0, \pi]$

Solution to Exercise 1.10

Observe that f has domain $\mathbb{C} - \{0\}$ and g has domain $\mathbb{C} - \{1\}$.

(a) $f + g$ has domain $\mathbb{C} - \{0, 1\}$ and rule

$$\begin{aligned}(f+g)(z) &= \frac{z-1}{z} + \frac{z}{z-1} \\ &= \frac{2z^2 - 2z + 1}{z(z-1)}.\end{aligned}$$

(b) $3f$ has domain $\mathbb{C} - \{0\}$ and $2ig$ has domain $\mathbb{C} - \{1\}$; hence $3f - 2ig$ has domain $\mathbb{C} - \{0, 1\}$ and rule

$$\begin{aligned}(3f - 2ig)(z) &= \frac{3(z-1)}{z} - \frac{2iz}{z-1} \\ &= \frac{3z^2 - 6z + 3 - 2iz^2}{z(z-1)} \\ &= \frac{(3-2i)z^2 - 6z + 3}{z(z-1)}.\end{aligned}$$

(c) fg has domain $\mathbb{C} - \{0, 1\}$ and rule

$$\begin{aligned}(fg)(z) &= \frac{z-1}{z} \times \frac{z}{z-1} \\ &= 1.\end{aligned}$$

(d) f/g has domain

$$(\mathbb{C} - \{0, 1\}) - \{0\} = \mathbb{C} - \{0, 1\}$$

and rule

$$\begin{aligned}(f/g)(z) &= \frac{z-1}{z} \bigg/ \frac{z}{z-1} \\ &= \left(\frac{z-1}{z}\right)^2.\end{aligned}$$

Solution to Exercise 1.11

(a) The domain of $f \circ g$ is

$$\begin{aligned}&\text{domain of } g - \left\{z : \frac{z}{z-1} \notin \mathbb{C} - \{0\}\right\} \\ &= (\mathbb{C} - \{1\}) - \left\{z : \frac{z}{z-1} = 0\right\} \\ &= (\mathbb{C} - \{1\}) - \{0\} = \mathbb{C} - \{0, 1\}.\end{aligned}$$

The rule of $f \circ g$ is

$$\begin{aligned}f(g(z)) &= \frac{\left(\dfrac{z}{z-1}\right) - 1}{\left(\dfrac{z}{z-1}\right)} \\ &= 1 - \left(\frac{z-1}{z}\right) \\ &= \frac{1}{z}.\end{aligned}$$

(b) The domain of $g \circ f$ is

$$\begin{aligned}&\text{domain of } f - \left\{z : \frac{z-1}{z} \notin \mathbb{C} - \{1\}\right\} \\ &= (\mathbb{C} - \{0\}) - \left\{z : \frac{z-1}{z} = 1\right\} \\ &= (\mathbb{C} - \{0\}) - \varnothing = \mathbb{C} - \{0\}.\end{aligned}$$

The rule of $g \circ f$ is

$$\begin{aligned}g(f(z)) &= \frac{\left(\dfrac{z-1}{z}\right)}{\left(\dfrac{z-1}{z}\right) - 1} \\ &= \frac{z-1}{z-1-z} \\ &= 1 - z.\end{aligned}$$

(c) The domain of $f \circ f$ is

$$\begin{aligned}&\text{domain of } f - \left\{z : \frac{z-1}{z} \notin \mathbb{C} - \{0\}\right\} \\ &= (\mathbb{C} - \{0\}) - \{z : z - 1 = 0\} \\ &= \mathbb{C} - \{0, 1\}.\end{aligned}$$

The rule of $f \circ f$ is

$$f(f(z)) = \frac{\left(\dfrac{z-1}{z}\right) - 1}{\left(\dfrac{z-1}{z}\right)} = 1 - \left(\frac{z}{z-1}\right) = \frac{1}{1-z}.$$

Solution to Exercise 1.12

The functions in parts (a), (b) and (c) are one-to-one, as we now show. We already know their image sets from Exercise 1.9.

(a) For each $w \in \mathbb{C}$, the image set of f, we wish to solve the equation

$$w = 2z + 1$$

to obtain a unique solution z in $\mathbb{C}$, the domain of f. This is achieved by the rearrangement

$$z = (w-1)/2.$$

Thus f is a one-to-one function, with image set $\mathbb{C}$. Hence f has an inverse function f^{-1} with domain $\mathbb{C}$ and rule

$$f^{-1}(w) = (w-1)/2.$$

(b) For each $w \in \mathbb{C} - \{0\}$, the image set of f, we wish to solve the equation

$$w = 1/(z-1)$$

to obtain a unique solution z in $\mathbb{C} - \{1\}$, the domain of f. This is achieved by the rearrangement

$$z = (1+w)/w.$$

Thus f is a one-to-one function, with image set $\mathbb{C} - \{0\}$. Hence f has an inverse function f^{-1} with domain $\mathbb{C} - \{0\}$ and rule

$$f^{-1}(w) = (1+w)/w.$$

(c) For each $w \in \mathbb{C} - \{1\}$, the image set of f, we wish to solve the equation

$$w = z/(z-1)$$

to obtain a unique solution z in $\mathbb{C} - \{1\}$, the domain of f. This is achieved by the rearrangement

$$z = w/(w-1).$$

Thus f is a one-to-one function, with image set $\mathbb{C} - \{1\}$. Hence f has an inverse function f^{-1} with domain $\mathbb{C} - \{1\}$ and rule

$$f^{-1}(w) = w/(w-1).$$

(Note that $f^{-1} = f$ in this case.)

(d) This function is not one-to-one, because (for example)

$$f(2) = |2-1| = 1 \quad \text{and}$$
$$f(0) = |0-1| = 1.$$

(e) This function is not one-to-one, because (for example)

$$f(i) = \operatorname{Re}(i+i) = 0 \quad \text{and}$$
$$f(0) = \operatorname{Re}(0+i) = 0.$$

(f) This function is not one-to-one, because (for example)

$$f(i) = |\operatorname{Arg} i| = |\pi/2| = \pi/2 \quad \text{and}$$
$$f(-i) = |\operatorname{Arg}(-i)| = |-\pi/2| = \pi/2.$$

Solution to Exercise 1.13

Let us first determine the image set of f, which is

$$\begin{aligned} f(A) &= \{z^3 : z \in A\} \\ &= \{0\} \cup \{w = z^3 : -\pi/3 < \operatorname{Arg} z \le \pi/3\}. \end{aligned}$$

By writing $z = r(\cos\theta + i\sin\theta)$, we see that $f(A)$ is equal to the union of $\{0\}$ and

$$\{w = r^3(\cos 3\theta + i\sin 3\theta) : r > 0,\ -\pi/3 < \theta \le \pi/3\}.$$

Let $\rho = r^3$ and $\phi = 3\theta$; then $f(A)$ is the union of $\{0\}$ and

$$\{w = \rho(\cos\phi + i\sin\phi) : \rho > 0,\ -\pi < \phi \le \pi\},$$

so $f(A) = \mathbb{C}$.

Now, for each $w \in \mathbb{C}$ we wish to solve the equation

$$w = z^3 \tag{S2}$$

to obtain a unique solution z in A. If $w = 0$, then equation (S2) has the unique solution $z = 0$. If $w \neq 0$, then w can be written in the form

$$w = \rho(\cos\phi + i\sin\phi),$$

where $\rho > 0$ and $-\pi < \phi \leq \pi$, and equation (S2) then has exactly three solutions:

$$z_0 = \rho^{1/3}\left(\cos\frac{\phi}{3} + i\sin\frac{\phi}{3}\right),$$
$$z_1 = \rho^{1/3}\left(\cos\left(\frac{\phi}{3} + \frac{2\pi}{3}\right) + i\sin\left(\frac{\phi}{3} + \frac{2\pi}{3}\right)\right),$$
$$z_2 = \rho^{1/3}\left(\cos\left(\frac{\phi}{3} + \frac{4\pi}{3}\right) + i\sin\left(\frac{\phi}{3} + \frac{4\pi}{3}\right)\right),$$

by Theorem 3.1 of Unit A1. Clearly, $z_0 \in A$, since $-\pi/3 < \phi/3 \leq \pi/3$, whereas z_1 and z_2 are not in A.

Thus f is a one-to-one function, with image set $\mathbb{C}$. Hence f has an inverse function f^{-1} with domain $\mathbb{C}$ and rule given by $f^{-1}(0) = 0$ and

$$f^{-1}(w) = \rho^{1/3}(\cos(\phi/3) + i\sin(\phi/3)),$$

where $w = \rho(\cos\phi + i\sin\phi)$, $\rho > 0$, $-\pi < \phi \leq \pi$.

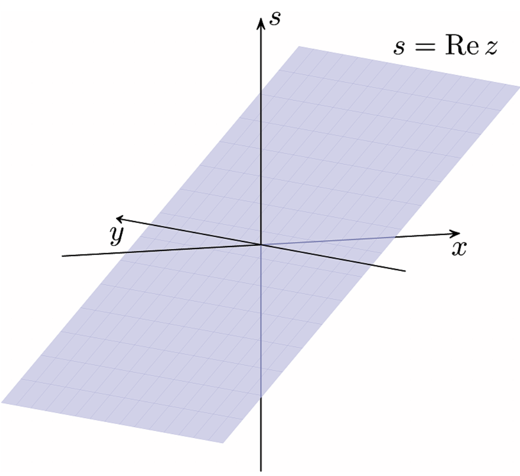
(Observe that for $w \neq 0$, $-\pi < \phi \leq \pi$, we have $\phi = \operatorname{Arg} w$, so z_0 is the principal cube root of w, namely $w^{1/3}$. Also, $0^{1/3} = 0$, by definition, so

$$f^{-1}(w) = w^{1/3} \quad (w \in \mathbb{C}).)$$

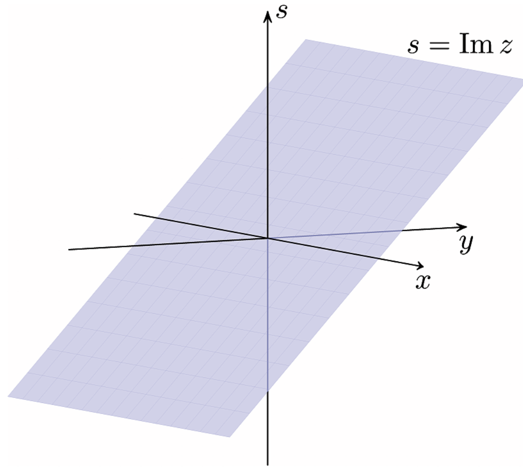
Solution to Exercise 2.1

Observe that in each of the following sketches we have rotated the x- and y-axes about the s-axis to help illustrate the shape of the surface.

(a) The surface with equation $s = \operatorname{Re} z$ is the plane that contains the y-axis and any line given by $s = x$ and $y = c$, for some constant c.



(b) The surface with equation $s = \operatorname{Im} z$ is the plane that contains the x-axis and any line given by $s = y$ and $x = c$, for some constant c.



Solution to Exercise 2.2

For $z \in \mathbb{C} - \{0\}$,

$$f(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}.$$

So

$$\operatorname{Re} f\colon z \longmapsto \frac{x}{x^2+y^2} \quad (z \in \mathbb{C} - \{0\})$$

and

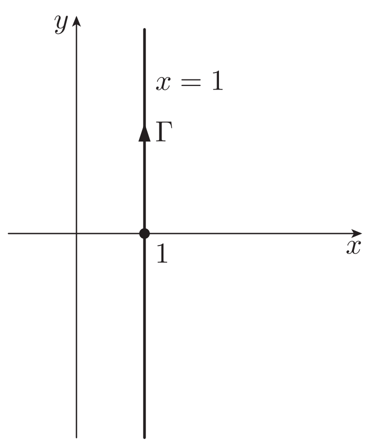
$$\operatorname{Im} f\colon z \longmapsto \frac{-y}{x^2+y^2} \quad (z \in \mathbb{C} - \{0\}).$$

Solution to Exercise 2.3

(a) Since $\gamma(t) = 1 + it$ $(t \in \mathbb{R})$, we have

$$x = 1, \quad y = t.$$

Hence Γ is the line with equation $x = 1$, as shown.



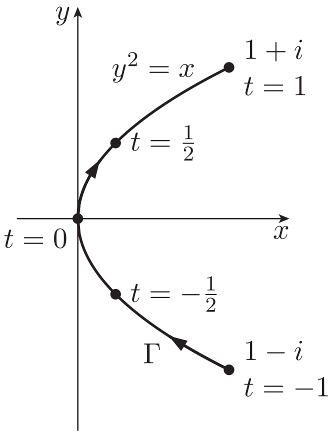
(b) Since $\gamma(t) = t^2 + it$ $(t \in [-1, 1])$, we have

$$x = t^2, \quad y = t.$$

A brief table of values is as follows.

t	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
x	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1
y	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1

Hence the path Γ is as shown.



Eliminating t from the equations $x = t^2$, $y = t$, we obtain

$$y^2 = x,$$

the equation of a parabola.

(c) Since $\gamma(t) = 1 - t + it$ $(t \in [0,1])$, we have

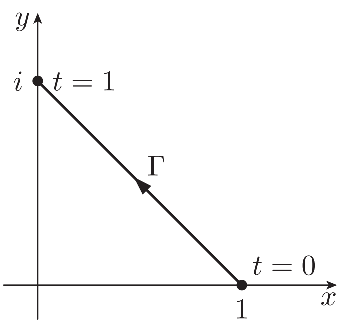
$$x = 1 - t, \quad y = t.$$

Eliminating t from these equations, we obtain

$$y = 1 - x,$$

the equation of a line.

When $t = 0$, we have $x = 1$, $y = 0$; when $t = 1$, we have $x = 0$, $y = 1$. Hence the path Γ is as shown.



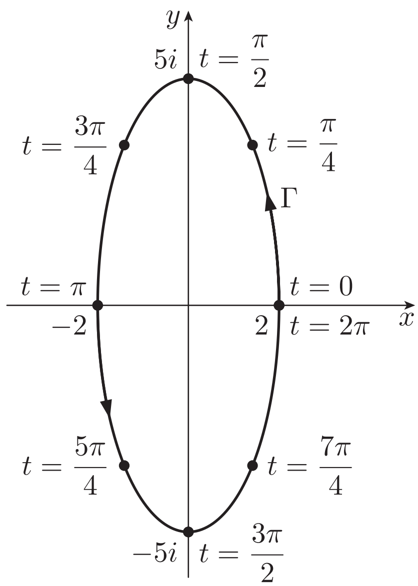
(d) Since $\gamma(t) = 2\cos t + 5i \sin t$ $(t \in [0, 2\pi])$, we have

$$x = 2\cos t, \quad y = 5 \sin t.$$

A brief table of values is as follows.

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
x	2	$\sqrt{2}$	0	$-\sqrt{2}$	-2	$-\sqrt{2}$	0	$\sqrt{2}$	2
y	0	$\frac{5}{\sqrt{2}}$	5	$\frac{5}{\sqrt{2}}$	0	$\frac{-5}{\sqrt{2}}$	-5	$\frac{-5}{\sqrt{2}}$	0

Hence the path Γ is as shown.



Eliminating t from the equations $x = 2\cos t$, $y = 5 \sin t$, we obtain

$$\frac{x^2}{4} + \frac{y^2}{25} = 1,$$

the equation of an ellipse.

Solution to Exercise 2.4

In each case we use the table of standard parametrisations.

(a) $\gamma(t) = (1 - t)(-2) + ti$
$\quad = 2(t - 1) + it \quad (t \in \mathbb{R}).$

Hence

$$x = 2(t - 1), \quad y = t.$$

(b) $\gamma(t) = (1 - t)(1) + t(1 + i)$
$\quad = 1 + ti \quad (t \in [0, 1]).$

Hence

$$x = 1, \quad y = t.$$

(c) $\gamma(t) = (1+i) + 1(\cos t + i\sin t)$
$$= 1 + \cos t + (1 + \sin t)i \quad (t \in [0, 2\pi]).$$
Hence
$$x = 1 + \cos t, \quad y = 1 + \sin t.$$

(d) The parabola is in standard form with $a = \frac{1}{4}$, so the standard parametrisation is
$$\gamma(t) = \tfrac{1}{4}t^2 + \tfrac{1}{2}it \quad (t \in \mathbb{R}).$$
Hence
$$x = \tfrac{1}{4}t^2, \quad y = \tfrac{1}{2}t.$$
(Of course, you may feel that the parametrisation
$$\gamma(t) = t^2 + it \quad (t \in \mathbb{R})$$
is simpler!)

Solution to Exercise 2.5

(a) Since $f(z) = \overline{z} = x - iy$, we have
$$\operatorname{Re} f\colon z \longmapsto x \qquad (z \in \mathbb{C}),$$
$$\operatorname{Im} f\colon z \longmapsto -y \quad (z \in \mathbb{C}).$$

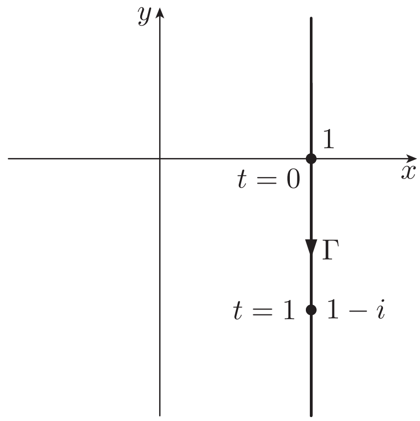
(b) Since $f(z) = iz = -y + ix$, we have
$$\operatorname{Re} f\colon z \longmapsto -y \quad (z \in \mathbb{C}),$$
$$\operatorname{Im} f\colon z \longmapsto x \qquad (z \in \mathbb{C}).$$

(c) Since
$$\begin{aligned} f(z) &= z^3 \\ &= (x+iy)^3 \\ &= x^3 - 3xy^2 + i(3x^2y - y^3), \end{aligned}$$
we have
$$\operatorname{Re} f\colon z \longmapsto x^3 - 3xy^2 \quad (z \in \mathbb{C}),$$
$$\operatorname{Im} f\colon z \longmapsto 3x^2y - y^3 \quad (z \in \mathbb{C}).$$

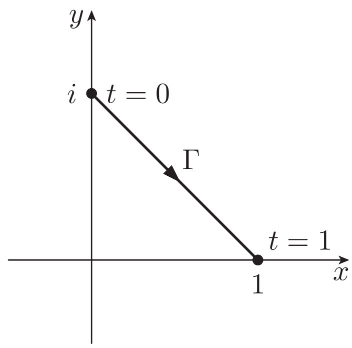
(d) Since $f(z) = |z| = \sqrt{x^2+y^2}$, we have
$$\operatorname{Re} f\colon z \longmapsto \sqrt{x^2+y^2} \quad (z \in \mathbb{C}),$$
$$\operatorname{Im} f\colon z \longmapsto 0 \qquad (z \in \mathbb{C}).$$

Solution to Exercise 2.6

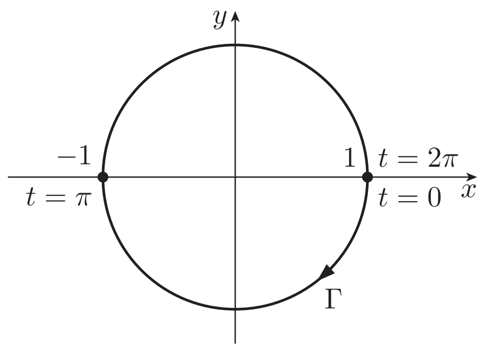
(a) $\gamma(t) = 1 - it \quad (t \in \mathbb{R})$



(b) $\gamma(t) = i + (1-i)t \quad (t \in [0,1])$



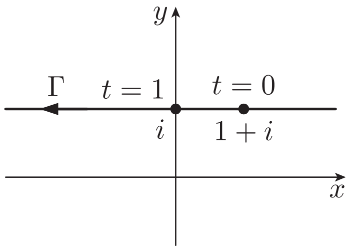
(c) $\gamma(t) = \cos t - i\sin t$
$$= \cos(-t) + i\sin(-t) \quad (t \in [0, 2\pi])$$



Solution to Exercise 2.7

(a) $\gamma(t) = (1-t)(1+i) + ti$
$$= 1 - t + i;$$
so the parametric equations are
$$x = 1 - t, \quad y = 1 \quad (t \in \mathbb{R}).$$

The path Γ is the line $y = 1$.

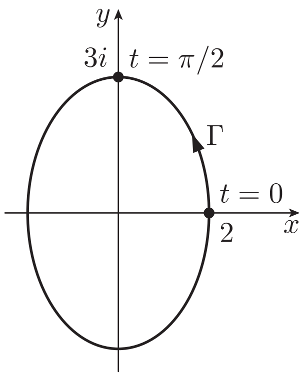


(b) From the table of standard parametrisations,

$$\gamma(t) = 2\cos t + 3i\sin t \quad (t \in [0, 2\pi])$$

is a parametrisation of the ellipse

$$\frac{x^2}{4} + \frac{y^2}{9} = 1.$$



(c) Since $\gamma(t) = 1 + 2\cos t - (1 - 2\sin t)i$, the parametric equations are

$$x = 1 + 2\cos t, \quad y = -1 + 2\sin t,$$

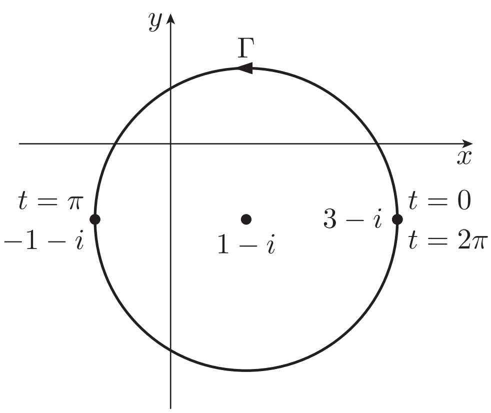
where $t \in [0, 2\pi]$. Hence

$$(x-1)^2 = 4\cos^2 t \quad \text{and} \quad (y+1)^2 = 4\sin^2 t,$$

so

$$(x-1)^2 + (y+1)^2 = 4.$$

This is the equation of the circle with radius 2 and centre $1 - i$.



Solution to Exercise 2.8

(a) $\gamma(t) = 1 - i + 3(\cos t + i\sin t) \quad (t \in [0, 2\pi])$.

(b) The equation $2x^2 + 3y^2 = 6$ is equivalent to

$$\frac{x^2}{3} + \frac{y^2}{2} = 1,$$

for which the standard parametrisation is

$$\gamma(t) = \sqrt{3}\cos t + i\sqrt{2}\sin t \quad (t \in [0, 2\pi]).$$

(c) The equation $8y^2 = x$ is equivalent to

$$y^2 = \tfrac{1}{8}x,$$

for which the standard parametrisation is

$$\gamma(t) = \tfrac{1}{32}t^2 + \tfrac{1}{16}it \quad (t \in \mathbb{R}).$$

Solution to Exercise 2.9

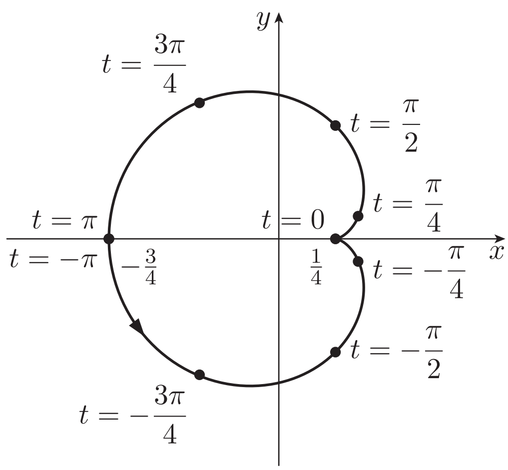
We have

$$\begin{aligned}\gamma(t) &= \tfrac{1}{2}(\cos t + i\sin t) - \tfrac{1}{4}(\cos 2t + i\sin 2t)\\ &= \tfrac{1}{2}\cos t - \tfrac{1}{4}\cos 2t + i\left(\tfrac{1}{2}\sin t - \tfrac{1}{4}\sin 2t\right),\end{aligned}$$

where $t \in [-\pi, \pi]$. Hence the table of values is as follows (where each non-zero value of x and y is given to two decimal places).

t	0	$\pm\frac{\pi}{4}$	$\pm\frac{\pi}{2}$	$\pm\frac{3\pi}{4}$	$\pm\pi$
x	0.25	0.35	0.25	−0.35	−0.75
y	0	±0.10	±0.50	±0.60	0

Plotting these points, we obtain the following rough sketch of the path (which is a curve called a cardioid).



Since

$$x = \tfrac{1}{2}\cos t - \tfrac{1}{4}\cos 2t, \quad y = \tfrac{1}{2}\sin t - \tfrac{1}{4}\sin 2t,$$

we have

$$\begin{aligned} x^2 + y^2 &= \left(\tfrac{1}{4}\cos^2 t - \tfrac{1}{4}\cos t\cos 2t + \tfrac{1}{16}\cos^2 2t\right) \\ &\quad + \left(\tfrac{1}{4}\sin^2 t - \tfrac{1}{4}\sin t\sin 2t + \tfrac{1}{16}\sin^2 2t\right) \\ &= \tfrac{1}{4}(\cos^2 t + \sin^2 t) \\ &\quad - \tfrac{1}{4}(\cos t\cos 2t + \sin t\sin 2t) \\ &\quad + \tfrac{1}{16}(\cos^2 2t + \sin^2 2t) \\ &= \tfrac{1}{4} - \tfrac{1}{4}\cos(2t - t) + \tfrac{1}{16} \\ &= \tfrac{5}{16} - \tfrac{1}{4}\cos t \\ &= \tfrac{1}{16}(5 - 4\cos t). \end{aligned}$$

Hence

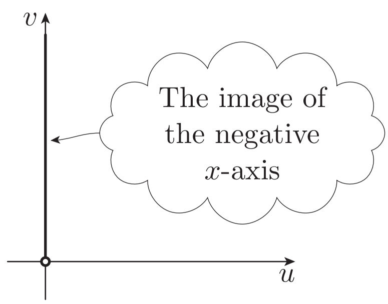
$$\begin{aligned} &4(x^2 + y^2)^2 - \tfrac{3}{2}(x^2 + y^2) + \tfrac{1}{2}x \\ &= \tfrac{1}{64}(5 - 4\cos t)^2 - \tfrac{3}{32}(5 - 4\cos t) \\ &\quad + \tfrac{1}{8}(2\cos t - \cos 2t) \\ &= \left(\tfrac{25}{64} - \tfrac{5}{8}\cos t + \tfrac{1}{4}\cos^2 t\right) - \left(\tfrac{15}{32} - \tfrac{3}{8}\cos t\right) \\ &\quad + \left(\tfrac{1}{4}\cos t - \tfrac{1}{8}(2\cos^2 t - 1)\right) \\ &= \tfrac{3}{64}, \end{aligned}$$

as required.

Solution to Exercise 2.10

The principal square root function $f(z) = \sqrt{z}$ maps $z = r(\cos\theta + i\sin\theta)$, where $\theta = \operatorname{Arg} z$, to $w = r^{1/2}(\cos\theta/2 + i\sin\theta/2)$.

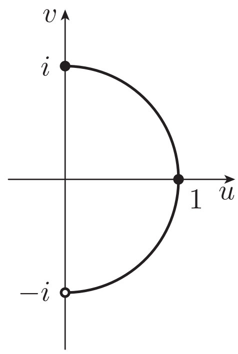
(a) If z is on the negative x-axis, then $\theta = \pi$, so its image w is such that $\operatorname{Arg} w = \pi/2$. Thus the negative real axis maps to the positive v-axis in the w-plane.



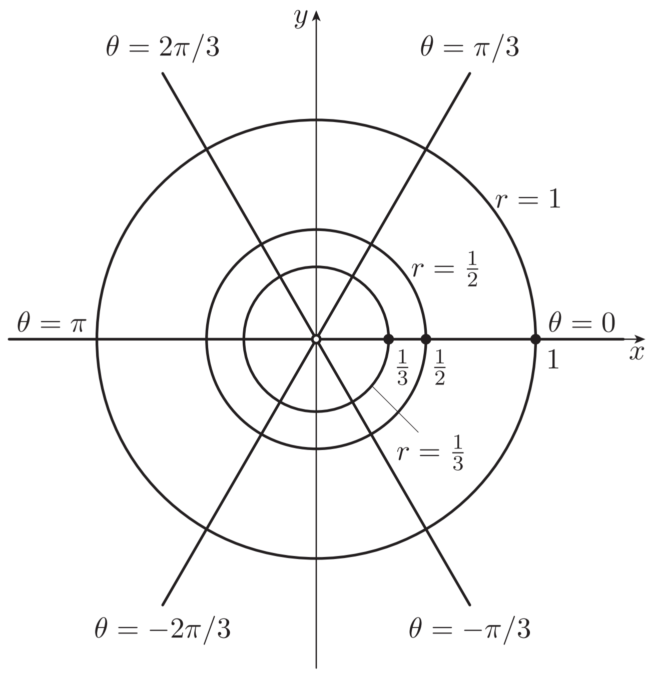
w-plane

(b) The image of the point z with modulus 1 and principal argument $\theta \in (-\pi, \pi]$ is the point w with modulus 1 and principal argument $\theta/2 \in (-\pi/2, \pi/2]$.

Hence the image of the circle $|z| = 1$ is the semicircle shown below, with one endpoint missing.



Solution to Exercise 3.1



Solution to Exercise 3.2

(a) $u = a - t, \quad v = a + t;$

adding these equations, we obtain

$$u + v = 2a.$$

(b) $u = a^2 - t^2, \quad v = 2at;$

hence

$$t^2 = \left(\frac{v}{2a}\right)^2,$$

so

$$u = a^2 - \frac{v^2}{4a^2},$$

which gives

$$v^2 = 4a^2(a^2 - u).$$

(c) $u = \dfrac{a}{a^2+t^2}, \quad v = \dfrac{-t}{a^2+t^2};$

squaring each of these expressions and adding the results, we obtain

$$u^2 = \frac{a^2}{(a^2+t^2)^2}, \quad v^2 = \frac{t^2}{(a^2+t^2)^2},$$

so

$$u^2+v^2 = \frac{a^2+t^2}{(a^2+t^2)^2} = \frac{1}{a^2+t^2} = \frac{u}{a}.$$

Solution to Exercise 3.3

The line $y = b$ has parametric equations

$$x = t, \quad y = b \quad (t \in \mathbb{R}).$$

Substituting these in

$$u = x - y, \quad v = x + y$$

(from equations (3.1)) gives the parametric equations of the image of the line $y = b$ under the function $f(z) = (1+i)z$. Thus

$$u = t - b, \quad v = t + b \quad (t \in \mathbb{R}).$$

Eliminating t, we obtain

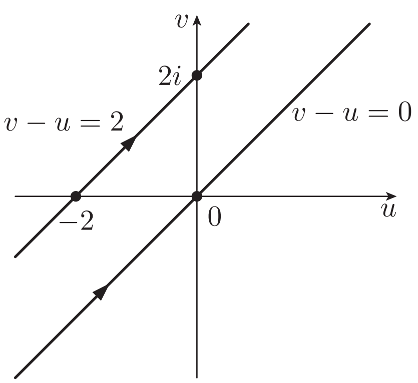
$$v - u = 2b,$$

which is the equation of a line.

The images of the lines $y = 1$ and $y = 0$ are, respectively,

$$v - u = 2 \quad \text{and} \quad v - u = 0.$$

They are shown below, as are the directions of increasing t. (The directions were not asked for in the exercise, but it is illuminating to include them anyway.)



Solution to Exercise 3.4

The line $y = b$ has parametric equations

$$x = t, \quad y = b \quad (t \in \mathbb{R}).$$

Substituting these in

$$u = x^2 - y^2, \quad v = 2xy$$

(from equations (3.2)) gives the parametric equations of the image of the line $y = b$ under the function $f(z) = z^2$. Thus

$$u = t^2 - b^2, \quad v = 2tb \quad (t \in \mathbb{R}).$$

Eliminating t, we obtain

$$v^2 = 4b^2(u + b^2), \quad b \neq 0,$$

which is the equation of a parabola.

When $b = 0$ we obtain the parametric equations

$$u = t^2, \quad v = 0 \quad (t \in \mathbb{R}),$$

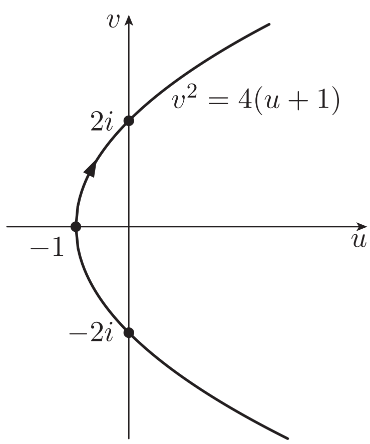
which are equations for the non-negative u-axis.

Therefore the images of the lines $y = 1$ and $y = 0$ are, respectively,

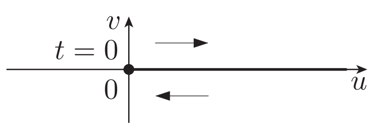
the parabola $v^2 = 4(u+1)$,

the non-negative u-axis, i.e. $v = 0$, $u \geq 0$.

They are shown below, as are the directions of increasing t.



The image of $y = 1$



The image of $y = 0$

Solution to Exercise 3.5

The line $y = b$ has parametric equations

$$x = t, \quad y = b \quad (t \in \mathbb{R}).$$

Substituting these in

$$u = \frac{x}{x^2 + y^2}, \quad v = \frac{-y}{x^2 + y^2}$$

(from equations (3.3)) gives the parametric equations of the image of the line $y = b$ under the function $f(z) = 1/z$. Thus

$$u = \frac{t}{t^2 + b^2}, \quad v = \frac{-b}{t^2 + b^2} \quad (t \in \mathbb{R}).$$

In the case $b \neq 0$, we obtain, on eliminating t,

$$u^2 + v^2 = -\frac{v}{b},$$

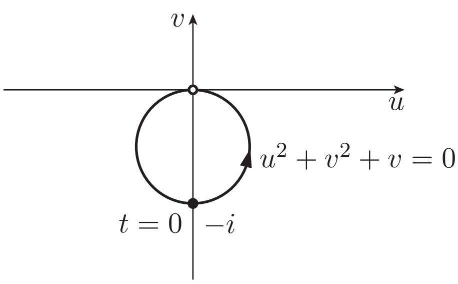
which is the equation of a circle through the origin, though the origin itself is excluded from the image.

When $b = 0$ we obtain the parametric equations

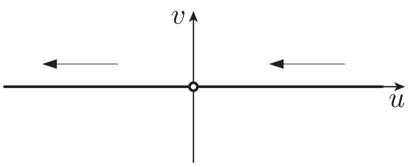
$$u = \frac{1}{t}, \quad v = 0 \quad (t \in \mathbb{R} - \{0\}),$$

which are parametric equations for the real axis, excluding the origin.

Therefore the images of the lines $y = 1$ and $y = 0$ are, respectively, the circle $u^2 + v^2 + v = 0$, excluding the origin, and the line $v = 0$, $u \neq 0$. They are shown below, as are the directions of increasing t.



The image of $y = 1$



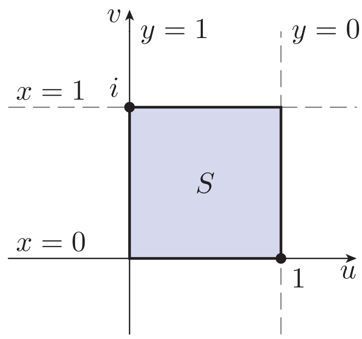
The image of $y = 0$

Solution to Exercise 3.6

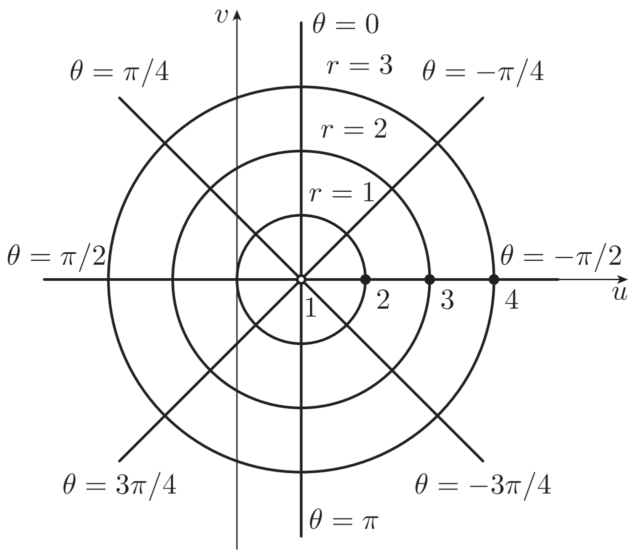
(a) The function $f(z) = iz + 1$ has the following geometric effect: it rotates the point z anticlockwise about the origin through $\pi/2$ and then translates the result to the right by one unit. Thus

- the image of the line $y = 0$ is the line $u = 1$
- the image of the line $x = 1$ is the line $v = 1$
- the image of the line $y = 1$ is the line $u = 0$
- the image of the line $x = 0$ is the line $v = 0$;

and the image of S is S (in the w-plane), as shown in the figure.

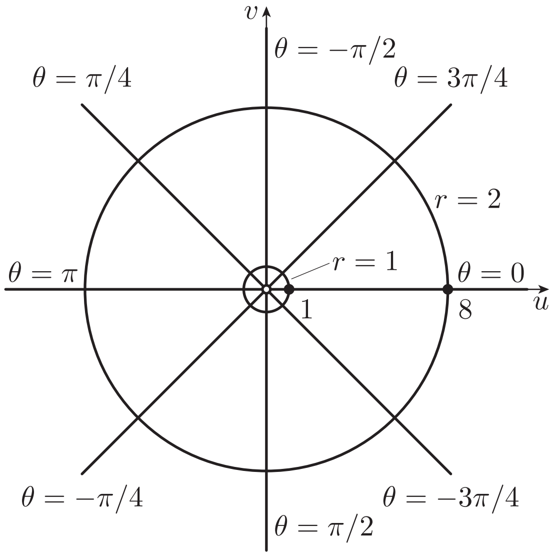


(b) Using the geometrical interpretation above, we obtain the image of the polar grid as follows.



Solution to Exercise 3.7

If z has modulus r and argument θ, then $w = f(z) = z^3$ has modulus r^3 and argument 3θ. Thus the image of the polar grid (with the circle $r = 3$ omitted) is as shown in the following figure.



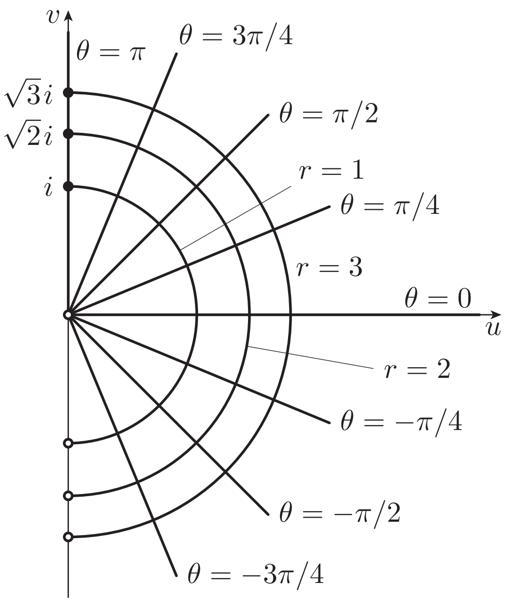
Solution to Exercise 3.8

If z has modulus r and principal argument θ, then $w = f(z) = \sqrt{z}$ has modulus $r^{1/2}$ and principal argument $\theta/2$. Thus

- the image of the ray $\theta = b$, where b is a constant in the interval $(-\pi, \pi]$, is the ray $\theta = b/2$
- the image of the circle with radius r and centre the origin is the semicircle (with one endpoint missing) given by

$$|w| = \sqrt{r}, \quad \theta \in (\pi/2, \pi/2].$$

Hence the image of the polar grid is as shown below.



Solution to Exercise 3.9

(a) The function $f(z) = z + i$ translates the point z one unit in the y-direction. The images of the Cartesian grid and the polar grid are shown below.

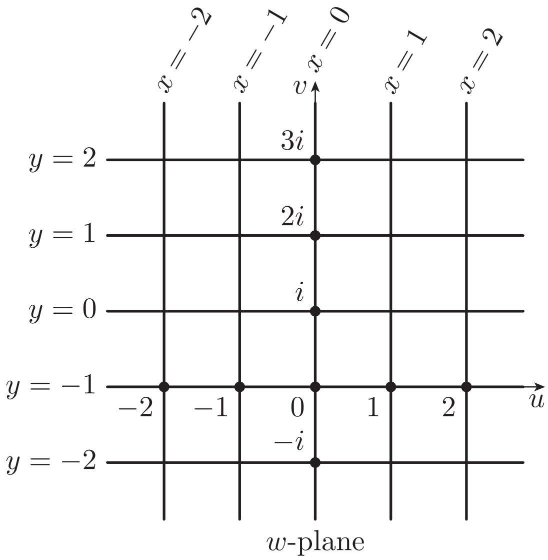


Image of Cartesian grid

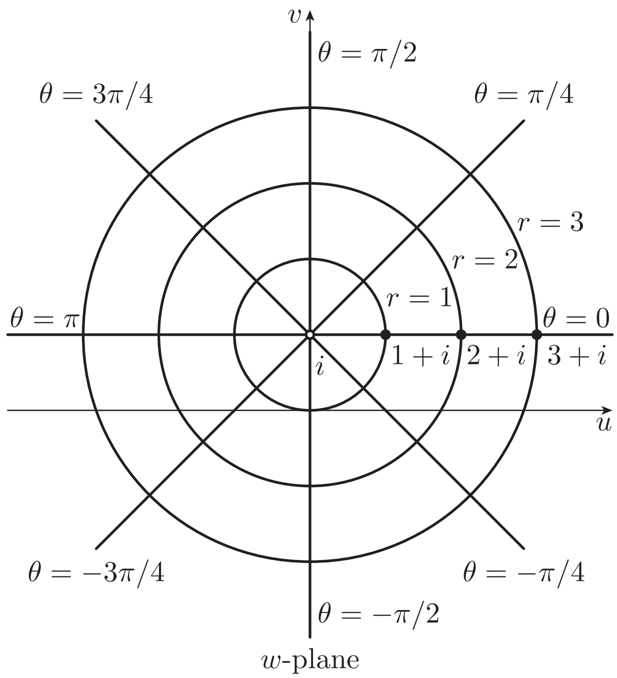


Image of polar grid

(b) The function $f(z) = 2z$ doubles the modulus of the point z, but leaves its argument unchanged. The images of the Cartesian grid and the polar grid are as follows.

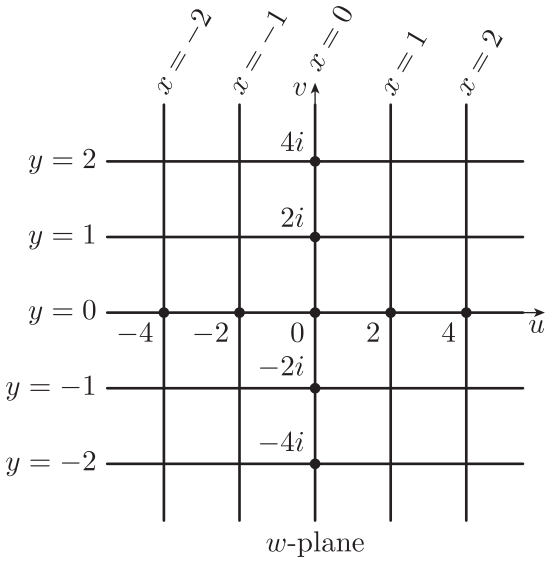


Image of Cartesian grid

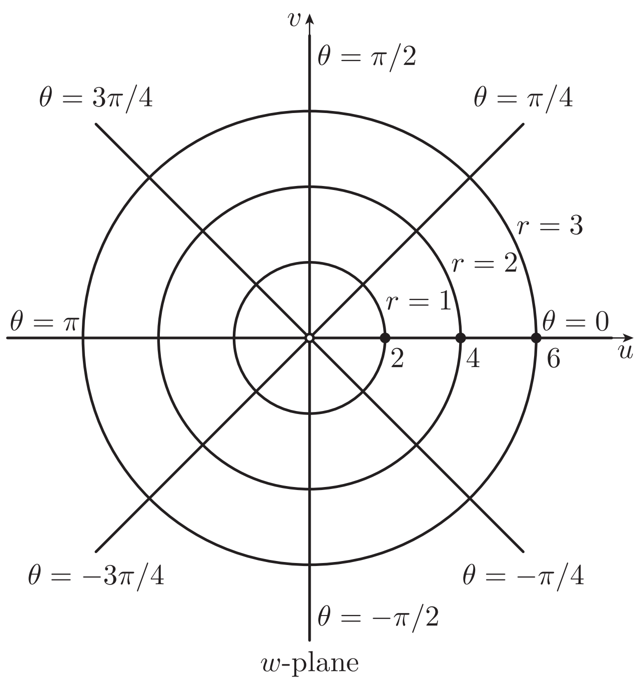


Image of polar grid

(c) The function $f(z) = 2 - iz$ rotates the point z about the origin through $\pi/2$ clockwise and then translates it 2 units to the right. Thus the images of the Cartesian grid and the polar grid are as follows.

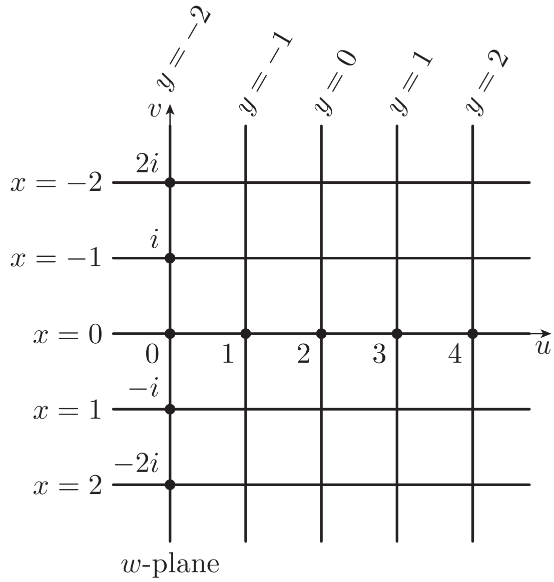


Image of Cartesian grid

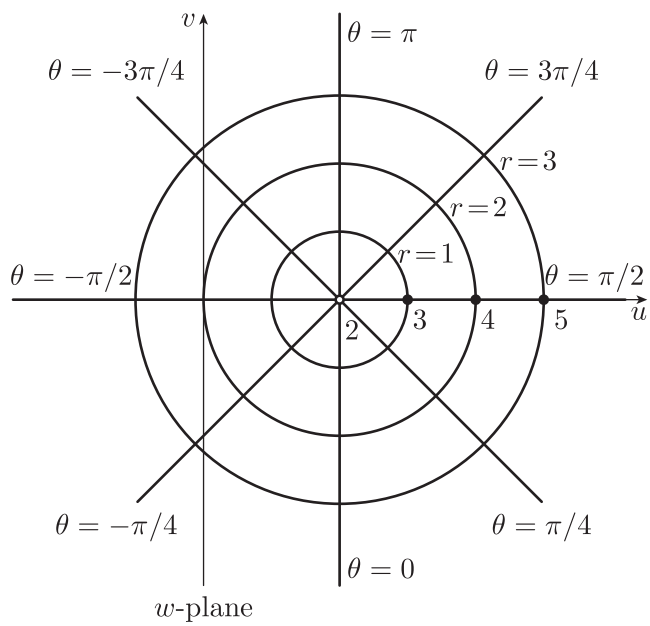


Image of polar grid

Alternatively, for the Cartesian grid we can use the parametric approach, as follows.

The image of z is

$$\begin{aligned} w = f(z) &= 2 - iz \\ &= 2 + y - ix, \end{aligned}$$

so

$$u = 2 + y, \quad v = -x. \tag{S3}$$

The line $x = a$ has parametric equations

$$x = a, \quad y = t \quad (t \in \mathbb{R}).$$

Substituting these in equations (S3) gives the parametric equations of the image,

$$u = 2 + t, \quad v = -a \quad (t \in \mathbb{R}),$$

which is the line $v = -a$.

Similarly, the line $y = b$ has parametric equations

$$x = t, \quad y = b \quad (t \in \mathbb{R}).$$

Substituting these in equations (S3) gives the parametric equations of the image,

$$u = 2 + b, \quad v = -t \quad (t \in \mathbb{R}),$$

which is the line $u = 2 + b$.

(d) Since $f(z) = iz^2 = i \times z^2$ and multiplication by i corresponds to rotation about the origin through $\pi/2$ anticlockwise, the images are found by rotating those in Figures 3.7 and 3.8 through $\pi/2$ anticlockwise.

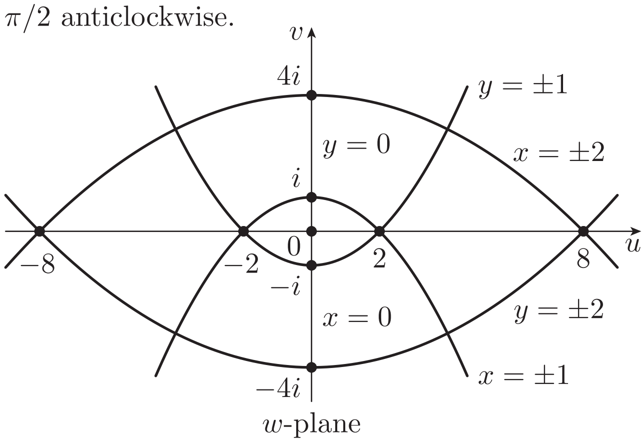


Image of Cartesian grid

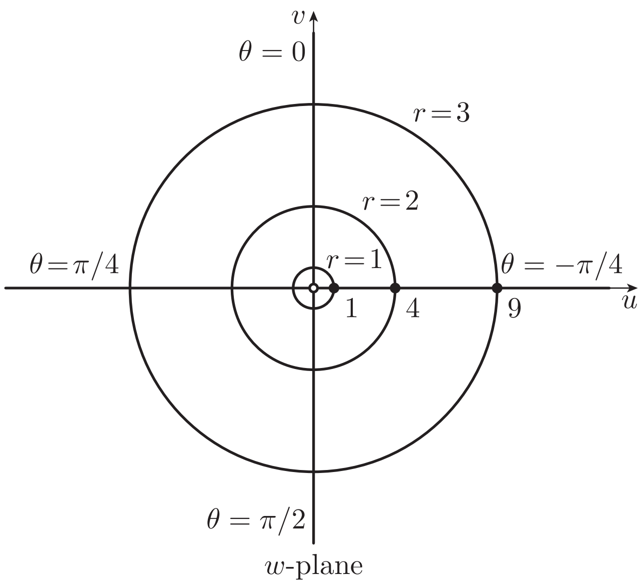


Image of polar grid

Solution to Exercise 4.1

(a) $e^{2\pi i} = e^0(\cos 2\pi + i \sin 2\pi) = 1$

(b) $e^{2+i\pi/3} = e^2\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$

$$= \frac{e^2}{2}(1 + \sqrt{3}i)$$

(c) $e^{-(1+i\pi)} = e^{-1}(\cos(-\pi) + i\sin(-\pi))$

$$= -1/e$$

Solution to Exercise 4.2

(a) $e^{z+2\pi i} = e^z e^{2\pi i}$ (Theorem 4.1(a))

$$= e^z,$$

because $e^{2\pi i} = 1$ (see Exercise 4.1(a)).

(b) $|e^z| = e^{\operatorname{Re} z}$ (Theorem 4.1(b))

$$\le e^{|z|},$$

because $\operatorname{Re} z \le |z|$ and $x \longmapsto e^x$ is an increasing function.

(c) The (complex) function exp is not one-to-one because $e^0 = e^{2\pi i} = 1$.

(d) **(i)** By equating moduli on each side of the equation $e^{x+iy} = 1$, we see that $e^x = 1$, and hence $e^{iy} = 1$ also. Therefore

$$\begin{aligned} e^{x+iy} = 1 &\iff e^x = 1 \text{ and } \cos y + i\sin y = 1 \\ &\iff x = 0 \text{ and } y = 2n\pi, \end{aligned}$$

where $n \in \mathbb{Z}$. It follows that

$$\{z : e^z = 1\} = \{2n\pi i : n \in \mathbb{Z}\}.$$

(ii) By equating moduli on each side of the equation $e^{x+iy} = -1$, we see that $e^x = 1$, and hence $e^{iy} = -1$. Therefore

$$\begin{aligned} e^{x+iy} = -1 &\iff e^x = 1 \text{ and } \cos y + i\sin y = -1 \\ &\iff x = 0 \text{ and } y = (2n+1)\pi, \end{aligned}$$

where $n \in \mathbb{Z}$. It follows that

$$\{z : e^z = -1\} = \{(2n+1)\pi i : n \in \mathbb{Z}\}.$$

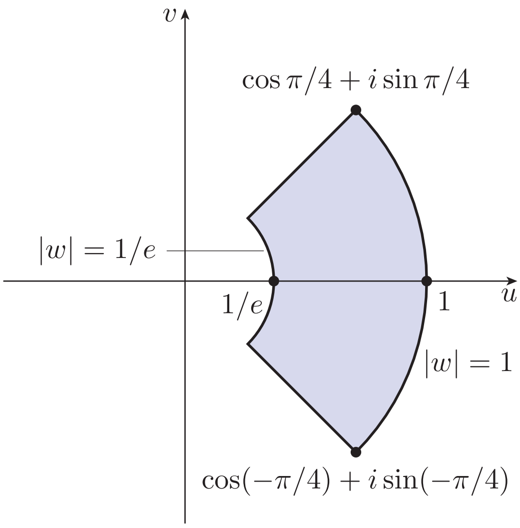
Solution to Exercise 4.3

Since $\sqrt{3} + i = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) = 2e^{i\pi/6}$, we have

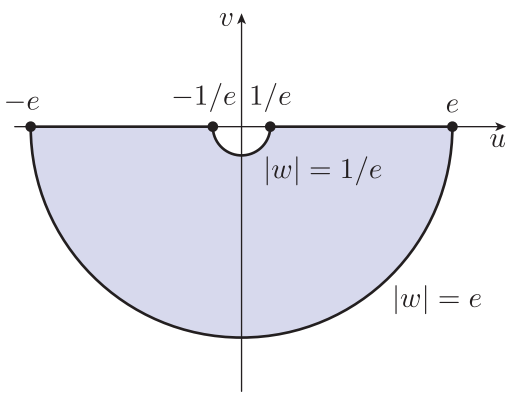
$$\begin{aligned} (\sqrt{3}+i)^{-6} &= (2e^{i\pi/6})^{-6} \\ &= 2^{-6}(e^{i\pi/6})^{-6} \\ &= 2^{-6}e^{-6\times i\pi/6} \\ &= 2^{-6}e^{-i\pi} = -1/64. \end{aligned}$$

Solution to Exercise 4.4

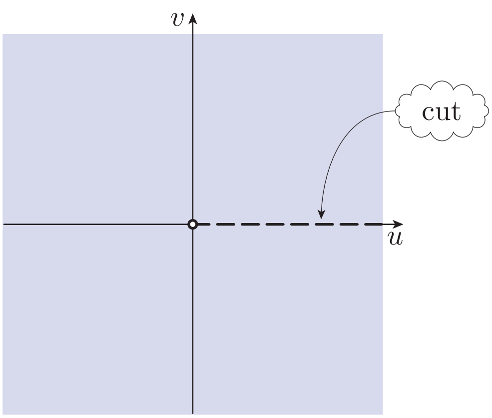
(a)



(b)



(c)



Solution to Exercise 4.5

(a)
$$\begin{aligned}\sin(\pi/2+i) &= \frac{1}{2i}\left(e^{i(\pi/2+i)} - e^{-i(\pi/2+i)}\right)\\ &= \frac{1}{2i}\left(e^{-1+i\pi/2} - e^{1-i\pi/2}\right)\\ &= \frac{1}{2i}\left(e^{-1}i - e(-i)\right)\\ &= \frac{1}{2}\left(e+e^{-1}\right)\end{aligned}$$

(b)
$$\begin{aligned}\cos i &= \frac{1}{2}\left(e^{i\times i} + e^{-i\times i}\right)\\ &= \frac{1}{2}\left(e+e^{-1}\right)\end{aligned}$$

(which equals $\sin(\pi/2+i)$, in fact).

Solution to Exercise 4.6

(a) (i)
$$\begin{aligned}\sin(-z) &= \frac{1}{2i}\left(e^{i(-z)} - e^{-i(-z)}\right)\\ &= \frac{1}{2i}\left(e^{-iz} - e^{iz}\right)\\ &= -\frac{1}{2i}\left(e^{iz} - e^{-iz}\right)\\ &= -\sin z\end{aligned}$$

(ii)
$$\begin{aligned}\cos(z+2\pi) &= \tfrac{1}{2}\left(e^{i(z+2\pi)} + e^{-i(z+2\pi)}\right)\\ &= \tfrac{1}{2}\left(e^{iz}e^{2\pi i} + e^{-iz}e^{-2\pi i}\right)\\ &= \tfrac{1}{2}\left(e^{iz} + e^{-iz}\right)\\ &= \cos z\end{aligned}$$

(b) (i)
$$\begin{aligned}\cos 2z &= \cos(z+z)\\ &= \cos z\cos z - \sin z\sin z\\ &\qquad \text{(Theorem 4.3(a))}\\ &= \cos^2 z - \sin^2 z\\ &= 2\cos^2 z - 1 \quad \text{(Theorem 4.3(b))}\end{aligned}$$

(ii)
$$\begin{aligned}&\tan(z_1 - z_2)\\ &= \tan(z_1 + (-z_2))\\ &= \frac{\tan z_1 + \tan(-z_2)}{1 - \tan z_1 \tan(-z_2)} \quad \text{(Theorem 4.3(a))}\\ &= \frac{\tan z_1 - \tan z_2}{1 + \tan z_1 \tan z_2} \quad \text{(Theorem 4.3(c))}\end{aligned}$$

Solution to Exercise 4.7

(a) By Theorem 4.3(a),

$$\begin{aligned}\cos z &= \cos(x+iy)\\ &= \cos x\cos(iy) - \sin x\sin(iy)\\ &= \cos x\cosh y - \sin x(i\sinh y)\\ &= \cos x\cosh y - i\sin x\sinh y.\end{aligned}$$

(b) Using part (a),

$$\begin{aligned}|\cos z|^2 &= \cos^2 x\cosh^2 y + \sin^2 x\sinh^2 y\\ &= \cos^2 x(1+\sinh^2 y) + \sin^2 x\sinh^2 y\\ &= \cos^2 x + \sinh^2 y\,(\cos^2 x + \sin^2 x)\\ &= \cos^2 x + \sinh^2 y.\end{aligned}$$

Solution to Exercise 4.8

(a) $$\begin{aligned}e^{3\pi i} &= e^0(\cos 3\pi + i\sin 3\pi)\\ &= -1\end{aligned}$$

(b) $$\begin{aligned}ee^{\pi i/2} &= e(\cos\pi/2 + i\sin\pi/2)\\ &= ei\end{aligned}$$

(c) $$\begin{aligned}e^{2\pi i/3} &= e^0(\cos 2\pi/3 + i\sin 2\pi/3)\\ &= -\frac{1}{2} + \frac{\sqrt{3}}{2}i\end{aligned}$$

(d) $$\begin{aligned}e^{-3\pi i/2} &= e^0(\cos(-3\pi/2) + i\sin(-3\pi/2))\\ &= i\end{aligned}$$

(e) $$\begin{aligned}e^{2+\pi i} &= e^2(\cos\pi + i\sin\pi)\\ &= -e^2\end{aligned}$$

(f) $$\begin{aligned}e^{3+\pi i/2} &= e^3(\cos\pi/2 + i\sin\pi/2)\\ &= e^3 i\end{aligned}$$

(g) $$\begin{aligned}e^{(\pi i/6)-1} &= e^{-1}(\cos\pi/6 + i\sin\pi/6)\\ &= \frac{\sqrt{3}}{2e} + \frac{1}{2e}i\end{aligned}$$

(h) $$e^{(\cos\theta + i\sin\theta)} = e^{\cos\theta}(\cos(\sin\theta) + i\sin(\sin\theta))$$

Solution to Exercise 4.9

(a) (i) We have

$$\left|\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right| = 1 \text{ and } \operatorname{Arg}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) = -\frac{\pi}{4},$$

so

$$\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} = 1e^{-\pi i/4} = e^{-\pi i/4}.$$

(ii) We have

$$|-1-i| = \sqrt{2} \text{ and } \operatorname{Arg}(-1-i) = -3\pi/4,$$

so

$$-1-i = \sqrt{2}e^{-3\pi i/4}.$$

(iii) We have

$$|1+\sqrt{3}i| = 2 \text{ and } \operatorname{Arg}(1+\sqrt{3}i) = \pi/3,$$

so

$$1+\sqrt{3}i = 2e^{\pi i/3}.$$

(b) (i) We have

$$\begin{aligned}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)^3 &= (e^{-\pi i/4})^3\\ &= e^{-3\pi i/4}\\ &= -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}.\end{aligned}$$

(ii) We have

$$\begin{aligned}(1+\sqrt{3}i)^{-7} &= (2e^{\pi i/3})^{-7}\\ &= 2^{-7}e^{-7\pi i/3}\\ &= 2^{-7}e^{-\pi i/3} \quad (\exp \text{ has period } 2\pi i)\\ &= 2^{-7}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\\ &= 2^{-8}(1-\sqrt{3}i).\end{aligned}$$

Solution to Exercise 4.10

(a) $$\begin{aligned}\sin(\pi+2i) &= \frac{1}{2i}(e^{i(\pi+2i)} - e^{-i(\pi+2i)})\\ &= \frac{1}{2i}(e^{-2+i\pi} - e^{2-i\pi})\\ &= \frac{1}{2i}(e^{-2}e^{i\pi} - e^2e^{-i\pi})\\ &= \frac{1}{2i}(-e^{-2} + e^2)\\ &= -i\left(\frac{e^2 - e^{-2}}{2}\right)\\ &= -i\sinh 2.\end{aligned}$$

Alternatively, using Theorems 4.3(a) and 4.4, we have

$$\begin{aligned}\sin(\pi+2i) &= \sin\pi\cos 2i + \cos\pi\sin 2i\\ &= 0 + (-1)\times(i\sinh 2)\\ &= -i\sinh 2.\end{aligned}$$

(b) $$\begin{aligned}\cos(\pi/2 - i) &= \tfrac{1}{2}(e^{i(\pi/2-i)} + e^{-i(\pi/2-i)})\\ &= \tfrac{1}{2}(e^{1+i\pi/2} + e^{-1-i\pi/2})\\ &= \tfrac{1}{2}(ee^{i\pi/2} + e^{-1}e^{-i\pi/2})\\ &= \tfrac{1}{2}(ei - e^{-1}i)\\ &= \left(\frac{e-e^{-1}}{2}\right)i\\ &= i\sinh 1.\end{aligned}$$

Alternatively, using Theorems 4.3(a) and 4.4, we have

$$\begin{aligned}\cos(\pi/2 - i) &= \cos\pi/2\cos i + \sin\pi/2\sin i\\ &= 0 + 1 \times i\sinh 1\\ &= i\sinh 1.\end{aligned}$$

(c) $$\begin{aligned}\tan i &= \frac{\sin i}{\cos i}\\ &= \frac{i\sinh 1}{\cosh 1} \quad \text{(Theorem 4.4)}\\ &= i\tanh 1\end{aligned}$$

Solution to Exercise 4.11

(a) Writing $z = x + iy$, and observing that e^x is real, we obtain

$$\begin{aligned}\overline{e^z} &= \overline{e^{x+iy}}\\ &= \overline{e^x e^{iy}}\\ &= e^x\overline{e^{iy}}\\ &= e^x(\cos y - i\sin y).\end{aligned}$$

Also,

$$\begin{aligned}e^{\overline{z}} &= e^{x-iy}\\ &= e^x e^{-iy}\\ &= e^x(\cos(-y) + i\sin(-y))\\ &= e^x(\cos y - i\sin y).\end{aligned}$$

Hence $\overline{e^z} = e^{\overline{z}}$.

(b) We have

$$\sin 2z = \frac{1}{2i}(e^{2iz} - e^{-2iz})$$

and

$$\begin{aligned}2\sin z\cos z &= 2 \times \left(\frac{e^{iz}-e^{-iz}}{2i}\right) \times \left(\frac{e^{iz}+e^{-iz}}{2}\right)\\ &= \frac{1}{2i}(e^{2iz} - e^{-2iz}).\end{aligned}$$

Hence $\sin 2z = 2\sin z\cos z$.

(c) Using part (a), we see that

$$\begin{aligned}\overline{\sin z} &= \overline{\frac{1}{2i}(e^{iz} - e^{-iz})}\\ &= \frac{1}{\overline{2i}}(\overline{e^{iz}} - \overline{e^{-iz}})\\ &= \frac{1}{2(-i)}(e^{\overline{iz}} - e^{\overline{-iz}})\\ &= -\frac{1}{2i}(e^{-i\overline{z}} - e^{i\overline{z}})\\ &= \frac{1}{2i}(e^{i\overline{z}} - e^{-i\overline{z}})\\ &= \sin\overline{z}.\end{aligned}$$

(d) We have

$$\begin{aligned}&\cosh(z_1 + z_2)\\ &= \cos(i(z_1+z_2)) \quad \text{(Theorem 4.4)}\\ &= \cos(iz_1 + iz_2)\\ &= \cos(iz_1)\cos(iz_2) - \sin(iz_1)\sin(iz_2)\\ &\qquad \text{(Theorem 4.3(a))}\\ &= \cosh z_1\cosh z_2 - (i\sinh z_1)(i\sinh z_2)\\ &\qquad \text{(Theorem 4.4)}\\ &= \cosh z_1\cosh z_2 + \sinh z_1\sinh z_2.\end{aligned}$$

(e) Using Theorem 4.4, we see that

$$\begin{aligned}\cosh^2 z - \sinh^2 z &= \cos^2(iz) - i^2\sin^2(iz)\\ &= \cos^2(iz) + \sin^2(iz) = 1,\end{aligned}$$

by Theorem 4.3(b).

Solution to Exercise 5.1

First we determine the image set of f:

$$\begin{aligned}f(A) &= \{e^z : z \in A\}\\ &= \{w = e^{x+iy} : x \in \mathbb{R},\ 0 \le y < 2\pi\}\\ &= \{w = e^x e^{iy} : x \in \mathbb{R},\ 0 \le y < 2\pi\}\\ &= \{w = \rho e^{i\phi} : \rho > 0,\ 0 \le \phi < 2\pi\}\\ &= \mathbb{C} - \{0\},\end{aligned}$$

where $\rho = e^x$ and $\phi = y$.

Now, for each $w \in \mathbb{C} - \{0\}$, we wish to solve the equation

$$w = e^z$$

to obtain a unique solution z in A. Each w in $\mathbb{C} - \{0\}$ can be written in the form

$$w = \rho e^{i\phi}, \quad \text{where } \rho > 0 \text{ and } 0 \le \phi < 2\pi,$$

and the equation $w = e^z$ is then

$$\rho e^{i\phi} = e^z = e^x e^{iy}, \quad \text{where } z = x + iy.$$

Thus, by equating moduli in the equation above, we see that x and y must satisfy

$$\rho = e^x \quad \text{and} \quad e^{i\phi} = e^{iy};$$

that is,

$$x = \log\rho \quad \text{and} \quad y = \phi + 2n\pi,$$

where $n \in \mathbb{Z}$. For $n = 0$, the solution is

$$z = x + iy = \log\rho + i\phi,$$

which lies in A, since $0 \leq \phi < 2\pi$, whereas the other solutions (with $n \neq 0$) lie outside A.

Thus f is a one-to-one function, with image set $\mathbb{C} - \{0\}$. Hence f has inverse function f^{-1} with domain $\mathbb{C} - \{0\}$ and rule

$$f^{-1}(w) = \log\rho + i\phi,$$

where $w = \rho e^{i\phi}$, $\rho > 0$, $0 \leq \phi < 2\pi$.

Solution to Exercise 5.2

(a) $\operatorname{Log} i = \log|i| + i\operatorname{Arg} i$

$$= \log 1 + i\frac{\pi}{2} = i\frac{\pi}{2}$$

(b) $\operatorname{Log}(\sqrt{3} - i) = \log|\sqrt{3} - i| + i\operatorname{Arg}(\sqrt{3} - i)$

$$= \log 2 - i\frac{\pi}{6}$$

(c) $\operatorname{Log}\left(\tfrac{1}{2} + \tfrac{1}{2}i\right) = \log\left|\tfrac{1}{2} + \tfrac{1}{2}i\right| + i\operatorname{Arg}\left(\tfrac{1}{2} + \tfrac{1}{2}i\right)$

$$= \log\frac{1}{\sqrt{2}} + i\frac{\pi}{4}$$

$$= -\log\sqrt{2} + i\frac{\pi}{4}$$

Solution to Exercise 5.3

(a) False. (The ellipse $4x^2 + 9y^2 = 1$ lies entirely inside the unit circle $|z| = 1$, so its image lies in the left half-plane.)

(b) False. (The ray $\theta = \pi/4$ lies inside, on and outside the unit circle $|z| = 1$, so its image is not confined to the right half-plane.)

(c) False. (Since $1 + 4i$ does not lie in the strip $\{w : -\pi < \operatorname{Im} w \leq \pi\}$, which is the image set of the function Log, there is no $z \in \mathbb{C}$ such that $\operatorname{Log} z = 1 + 4i$.)

(d) True. (Since

$$1 + \tfrac{1}{4}i \in \{w : -\pi < \operatorname{Im} w \leq \pi\},$$

there is a $z \in \mathbb{C}$ such that $\operatorname{Log} z = 1 + \frac{1}{4}i$.)

Solution to Exercise 5.4

(a)
$$\begin{aligned}
(1+i)^{2/3} &= \exp\left(\tfrac{2}{3}\operatorname{Log}(1+i)\right) \\
&= \exp\left(\tfrac{2}{3}(\log\sqrt{2} + i\pi/4)\right) \\
&= \exp\left(\tfrac{1}{3}\log 2 + i\pi/6\right) \\
&= 2^{1/3}e^{i\pi/6} \\
&= 2^{1/3}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) \\
&= 2^{-2/3}(\sqrt{3} + i)
\end{aligned}$$

(b)
$$\begin{aligned}
i^{1+i} &= \exp((1+i)\operatorname{Log} i) \\
&= \exp\left((1+i)i\frac{\pi}{2}\right) \quad \text{(Exercise 5.2(a))} \\
&= e^{-\pi/2}e^{i\pi/2} \\
&= e^{-\pi/2}i
\end{aligned}$$

Solution to Exercise 5.5

Since $z^\alpha = \exp(\alpha\operatorname{Log} z)$, we have

$$\begin{aligned}
z^{1/n} &= \exp\left(\frac{1}{n}\operatorname{Log} z\right) \\
&= \exp\left(\frac{1}{n}(\log|z| + i\operatorname{Arg} z)\right) \\
&= \exp\left(\frac{1}{n}(\log\rho + i\phi)\right) \\
&\qquad (\text{where } \rho = |z|,\ \phi = \operatorname{Arg} z) \\
&= \exp\left(\log\rho^{1/n} + i\frac{\phi}{n}\right) \\
&= \rho^{1/n}e^{i\phi/n} \\
&= \rho^{1/n}\left(\cos\frac{\phi}{n} + i\sin\frac{\phi}{n}\right),
\end{aligned}$$

which is the principal nth root of $z = \rho(\cos\phi + i\sin\phi)$ because ϕ is the principal argument of z (see Subsection 3.1 of Unit A1).

Solution to Exercise 5.6

(a) Consider $z_1 = -1 = z_2$ and $\alpha = \frac{1}{2}$. Then

$$\begin{aligned} z_1^\alpha z_2^\alpha &= (-1)^{1/2}(-1)^{1/2} \\ &= i \times i \quad \text{(Example 5.3(a))} \\ &= -1. \end{aligned}$$

However,

$$\begin{aligned} (z_1 z_2)^\alpha &= ((-1) \times (-1))^{1/2} \\ &= 1^{1/2} \\ &= 1, \end{aligned}$$

so $z_1^\alpha z_2^\alpha \neq (z_1 z_2)^\alpha$.

(b) By definition,

$$\begin{aligned} z^\alpha &= \exp(\alpha(\log|z| + i \operatorname{Arg} z)), \\ z^\beta &= \exp(\beta(\log|z| + i \operatorname{Arg} z)), \end{aligned}$$

so, using Theorem 4.1(a),

$$\begin{aligned} z^\alpha z^\beta &= e^{\alpha(\log|z| + i \operatorname{Arg} z)} e^{\beta(\log|z| + i \operatorname{Arg} z)} \\ &= e^{\alpha(\log|z| + i \operatorname{Arg} z) + \beta(\log|z| + i \operatorname{Arg} z)} \\ &= e^{(\alpha+\beta)(\log|z| + i \operatorname{Arg} z)} \\ &= z^{\alpha+\beta}, \end{aligned}$$

by definition.

Solution to Exercise 5.7

(a) $$\begin{aligned} \operatorname{Log}(-2) &= \log|-2| + i \operatorname{Arg}(-2) \\ &= \log 2 + i\pi \end{aligned}$$

(b) $$\begin{aligned} \operatorname{Log}(i^3) &= \operatorname{Log}(-i) \\ &= \log|-i| + i \operatorname{Arg}(-i) \\ &= -i\pi/2 \end{aligned}$$

(c) $$\begin{aligned} \operatorname{Log}(1+i) &= \log|1+i| + i \operatorname{Arg}(1+i) \\ &= \log\sqrt{2} + i\pi/4 \end{aligned}$$

(d) $$\begin{aligned} \operatorname{Log}\sqrt{3} &= \log\sqrt{3} + i \operatorname{Arg}\sqrt{3} \\ &= \log\sqrt{3} \end{aligned}$$

(e) $$\begin{aligned} \operatorname{Log}(i-\sqrt{3}) &= \log|i-\sqrt{3}| + i \operatorname{Arg}(i-\sqrt{3}) \\ &= \log 2 + i5\pi/6 \end{aligned}$$

(f) $$\begin{aligned} \operatorname{Log}\left(\frac{1-i}{\sqrt{2}}\right) &= \log\left|\frac{1-i}{\sqrt{2}}\right| + i \operatorname{Arg}\left(\frac{1-i}{\sqrt{2}}\right) \\ &= \log 1 + i(-\pi/4) \\ &= -i\pi/4 \end{aligned}$$

Solution to Exercise 5.8

(a) $$\begin{aligned} i^{-i} &= \exp(-i \operatorname{Log} i) \\ &= \exp(-i(0 + i\pi/2)) \\ &= e^{\pi/2} \end{aligned}$$

(b) $$\begin{aligned} (-i)^i &= \exp(i \operatorname{Log}(-i)) \\ &= \exp(i(0 - i\pi/2)) \\ &= e^{\pi/2} \end{aligned}$$

(c) $$\begin{aligned} (1-i)^i &= \exp(i \operatorname{Log}(1-i)) \\ &= \exp(i(\log|1-i| + i \operatorname{Arg}(1-i))) \\ &= \exp(i(\log\sqrt{2} + i(-\pi/4))) \\ &= \exp(i\log\sqrt{2} + \pi/4) \\ &= e^{\pi/4}(\cos(\log\sqrt{2}) + i\sin(\log\sqrt{2})) \end{aligned}$$

(d) $$\begin{aligned} (-1)^i &= \exp(i \operatorname{Log}(-1)) \\ &= \exp(i(0 + i\pi)) \\ &= e^{-\pi} \end{aligned}$$

Unit A3

Continuity

Introduction

In Unit A2 we introduced complex functions and began to study the way in which they map subsets of $\mathbb{C}$ into $\mathbb{C}$. In this unit we look more closely at such functions and prove that many of them have the property of being *continuous*.

Roughly speaking, a function is continuous if it always maps nearby points in the domain to nearby points in the codomain. More precisely, a function f is continuous if any *convergent sequence* $z_1, z_2, z_3, \ldots$ in the domain of f, with limit α (say), is mapped by f to a convergent sequence $f(z_1), f(z_2), f(z_3), \ldots$ in the codomain of f with limit $f(\alpha)$, as indicated in Figure 0.1.

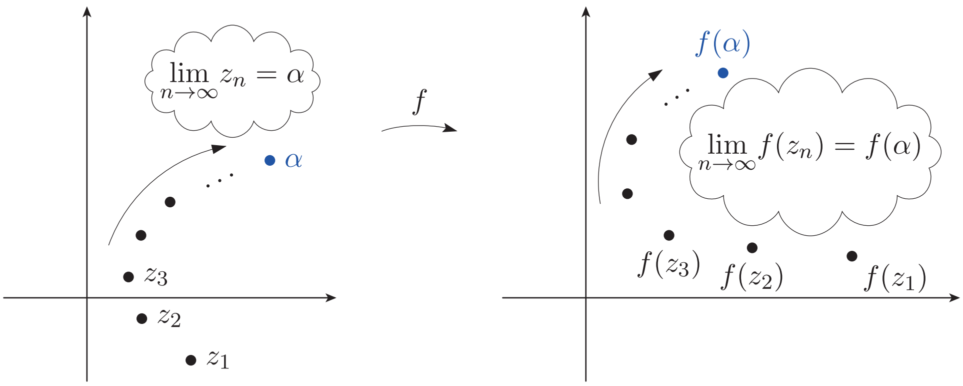


Figure 0.1 A convergent sequence mapped by f to another convergent sequence

Continuity is important in complex analysis because, in many cases, the easiest way to prove that a given function has a certain property is to use the fact that the function is continuous. For example, to prove that for a given polynomial function p there is a complex number β with $|\beta| \leq 1$ such that

$$|p(z)| \leq |p(\beta)|, \quad \text{whenever } |z| \leq 1,$$

we can make use of the fact that p is continuous and appeal to a general result about continuous functions called the Extreme Value Theorem (which you will meet in Section 5). In this case, the fact that p is continuous on the closed disc $\{z : |z| \leq 1\}$, together with the Extreme Value Theorem, tells us that such a point β *exists*. To actually *calculate* β, and calculate the maximum value of $|p(z)|$ on the disc, is, in general, a difficult task.

In Section 1 we define the notion of a convergent sequence and describe many properties of such sequences. We also discuss *divergent sequences*. In Section 2 we use the notion of a convergent sequence to define a continuous function in the way described above. We then give an alternative, equivalent way to define a continuous function, and we describe rules for combining and composing continuous functions. Using these rules,

together with a list of *basic continuous functions*, we can show that most of the functions introduced so far are continuous.

In Section 3 we define the *limit of a function*, a notion which is closely related to continuity; this will be an essential tool in Unit A4 when we come to discuss differentiability of complex functions.

In Sections 4 and 5 we discuss different types of subsets of $\mathbb{C}$ that we will need throughout the module: *open sets*, *connected sets*, *regions*, *closed sets*, *bounded sets* and *compact sets*.

Unit guide

This unit introduces many basic concepts that are essential tools in the study of complex analysis, and it also contains a larger than average number of proofs. Many of these proofs are short, but some are rather tricky, and we have tried to ease your study by indicating those proofs that may be omitted on a first reading. Once you are familiar with all the basic concepts, it should be much easier for you to follow these proofs.

1 Sequences

After working through this section, you should be able to:

- explain the statement 'the sequence (z_n) is convergent with limit α'
- recognise certain *basic null sequences*
- show that a sequence is null by working from the definition, and by using the Squeeze Rule
- use the Combination Rules for sequences
- explain the statement 'the sequence (z_n) tends to infinity'
- use the Subsequence Rules to recognise *divergent sequences*.

1.1 Convergent sequences

Ever since learning to count, you have been familiar with the sequence of natural numbers

$$1, 2, 3, 4, 5, 6, \ldots.$$

You will have also encountered many other sequences of numbers, such as

$$1, 3, 5, 7, 9, 11, \ldots,$$
$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \ldots.$$

In this section we study sequences of *complex* numbers. We begin with a definition and some notation.

Definitions

A **(complex) sequence** is an unending list of complex numbers

$$z_1, z_2, z_3, \ldots .$$

The complex number z_n is called the ***n*th term** of the sequence, and the sequence is denoted by (z_n).

Remarks

1. Note that a *real sequence* (a sequence of real numbers) is a particular type of complex sequence.
2. The sequences specified in this definition and exemplified below all have first term z_1. Sometimes it is convenient or necessary to start with a term other than z_1. For example, the sequence

   $$z_n = \frac{i}{n^2 - 1}, \quad n = 2, 3, \ldots,$$

   begins with z_2.

A sequence is often defined by stating an explicit formula for the nth term. This can be done in more than one way. For example, the expression (i^n) denotes the sequence

$$i, i^2, i^3, i^4, \ldots,$$

as does

$$z_n = i^n, \quad n = 1, 2, \ldots .$$

Although round brackets '(' and ')' are used for many purposes in mathematics, the meaning of notation such as (i^n) should always be clear from the context. (In other texts you may see the notation $\{i^n\}$ used in place of (i^n).)

When z_n is given by a complicated expression, it is usually best to use the second of the two ways above of representing a sequence. For example, you may find it clearer to write

$$z_n = (n + 1)^2, \quad n = 1, 2, \ldots,$$

rather than $((n + 1)^2)$.

It is often helpful to picture how a given sequence behaves by plotting the first few terms in the complex plane; two examples are shown in Figure 1.1.

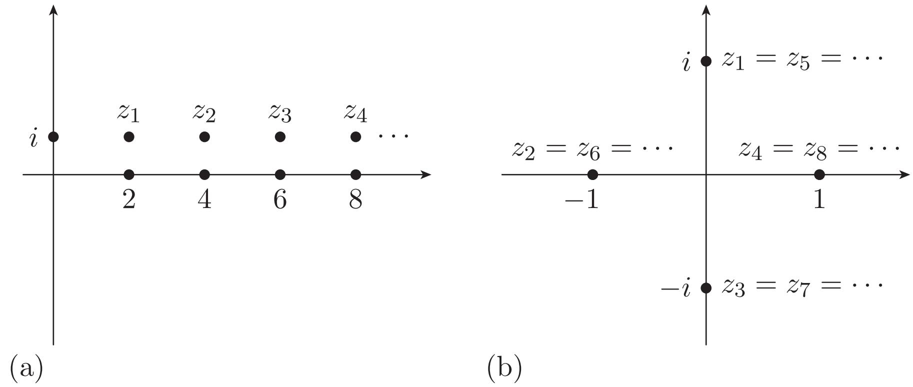


Figure 1.1 (a) $z_n = 2n + i,\ n = 1, 2, \ldots$ (b) $z_n = i^n,\ n = 1, 2, \ldots$

Exercise 1.1

Plot the first four terms of each of the following sequences.

(a) $z_n = i/n, \quad n = 1, 2, \ldots$ (b) $z_n = 1/n + in, \quad n = 1, 2, \ldots$

(c) $z_n = (2i)^n, \quad n = 1, 2, \ldots$

The terms of the sequence illustrated in Figure 1.1(a) lie on the line $y = 1$ and march out to the right as n increases; the terms of the sequence in Figure 1.1(b) go round and round the origin (on the circle $|z| = 1$) as n increases. These sequences do not appear to be convergent; that is, as n gets larger and larger, they do not settle down near any single point of $\mathbb{C}$. By contrast, consider the following sequence, whose first few terms (correct to two significant figures) are plotted in Figure 1.2:

$$z_n = (0.9i)^n, \quad n = 1, 2, \ldots.$$

n	1	2	3	4	5	6	7	8
z_n	$0.9i$	-0.81	$-0.73i$	0.66	$0.59i$	-0.53	$-0.48i$	0.43

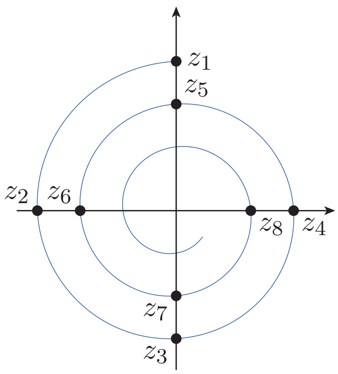


Figure 1.2 $z_n = (0.9i)^n$, $n = 1, 2, \ldots$

From this diagram, we observe that the sequence (z_n) spirals around the origin, getting closer and closer to it. Indeed, it seems likely that we can make the terms as close as we please to 0 by taking n large enough. More precisely, no matter how small an open disc centred on the origin we consider, the terms of the sequence (z_n) will eventually lie inside the disc. For example, if we take the radius of the disc to be 0.6, then

$$|z_n| < 0.6, \quad \text{for all } n > 4.$$

More generally, it appears that for each radius $\varepsilon > 0$, there is an integer N such that

$$|z_n| < \varepsilon, \quad \text{for all } n > N. \tag{1.1}$$

We will prove this statement later in the section – see Theorem 1.2(b).

In fact, statement (1.1) gives a precise definition of what it means for a sequence (z_n) to be convergent with limit 0. Moreover, this statement can

be readily adapted to give a precise definition of what it means for a given sequence (z_n) to be convergent with limit α: the terms of (z_n) must eventually lie in any open disc, centred at α, no matter how small its radius (where *eventually* means 'for all but a finite number of values of n').

Definitions

The sequence (z_n) is **convergent with limit α**, or **converges to α**, or **tends to α**, if for each positive number ε, there is an integer N such that

$$|z_n - \alpha| < \varepsilon, \quad \text{for all } n > N.$$

If (z_n) converges to α, then we write

- *either* $\lim_{n\to\infty} z_n = \alpha$
- *or* $z_n \to \alpha$ as $n \to \infty$.

If the limit α is 0, then (z_n) is called a **null sequence**.

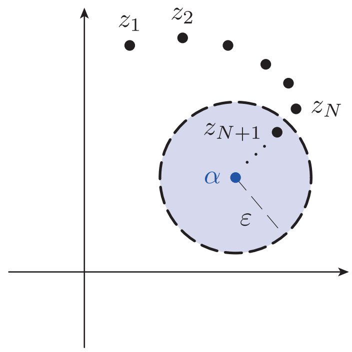


Figure 1.3 A convergent sequence

The definitions are illustrated in Figure 1.3.

Remarks

1. The statement '$\lim_{n\to\infty} z_n = \alpha$' is read as 'the limit of z_n as n tends to infinity is (equal to) α', and '$z_n \to \alpha$ as $n \to \infty$' is read as 'z_n tends to α as n tends to infinity'.
2. The Greek lower-case letter ε (epsilon) is used to denote a positive number, as is commonplace in real and complex analysis.
3. If a sequence (z_n) is convergent, then it has a *unique* limit. To see this, observe that if (z_n) has limits α and β with $\alpha \neq \beta$, and if we put $\varepsilon = \frac{1}{3}|\alpha - \beta|$, then the open discs

   $$\{z : |z - \alpha| < \varepsilon\} \quad \text{and} \quad \{z : |z - \beta| < \varepsilon\}$$

   do not overlap (Figure 1.4), and the terms z_n cannot eventually lie in both discs.
4. If a given sequence converges to α, then this remains true if we *add, delete or alter a finite number of terms*. Loosely speaking, 'a finite number of terms do not affect convergence'.

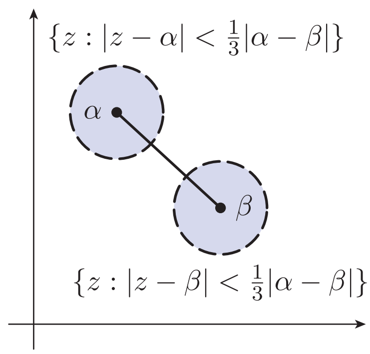


Figure 1.4 Two discs, centred at α and β, which do not overlap

We often have recourse to the following lemma, which follows immediately from the definition of convergence.

Lemma 1.1

The sequence (z_n) converges to α if and only if $(z_n - \alpha)$ is a null sequence. That is,

$$z_n \to \alpha \text{ as } n \to \infty \iff z_n - \alpha \to 0 \text{ as } n \to \infty.$$

Equivalently, (z_n) converges to α if and only if the real sequence $a_n = |z_n - \alpha|$, $n = 1, 2, \ldots$, is null.

A sequence (z_n) is said to be **constant** if there is a number α with $z_n = \alpha$, $n = 1, 2, \ldots$. Clearly, this sequence converges to α, because for each positive number ε, we have $|z_n - \alpha| = 0 < \varepsilon$, for $n = 1, 2, \ldots$.

You will need to be familiar with techniques for finding limits of convergent sequences. First we give examples of simple null sequences and then we discuss rules for dealing with more complicated sequences.

Example 1.1

Prove that the sequence $z_n = i/n$, $n = 1, 2, \ldots$, is null.

Solution

We need to show that for each positive number ε, there is an integer N such that

$$\left|\frac{i}{n}\right| < \varepsilon, \quad \text{for all } n > N. \tag{1.2}$$

But $|i/n| = 1/n$ and we know that

$$\frac{1}{n} < \varepsilon \iff n > \frac{1}{\varepsilon}.$$

Therefore statement (1.2) is true if we choose N to be any positive integer greater than $1/\varepsilon$ because, with this choice, if $n > N$, then $n > 1/\varepsilon$. Thus

$$\frac{1}{n} < \varepsilon, \quad \text{for all } n > N,$$

as required. Hence (z_n) is a null sequence.

For an example of how N is chosen in Example 1.1, suppose that $\varepsilon = 0.12$. Then $1/\varepsilon = 8.3$, to two significant figures, so we can choose N to be any integer greater than 8.3; say, $N = 9$. Then

$$\frac{1}{n} < 0.12, \quad \text{for all } n > 9.$$

Note that, in the solution to Example 1.1 and in general, the integer N becomes larger as the positive number ε becomes smaller, as you would expect.

Exercise 1.2

Prove that each of the following sequences is null.

(a) $z_n = 1/\sqrt{n}, \quad n = 1, 2, \ldots$ (b) $z_n = (1 + i)/n, \quad n = 1, 2, \ldots$

Proving that a sequence is null from the definition can be tricky, as illustrated above. We now introduce a result that enables us to avoid using

the definition in many cases. Consider, for example, the sequence

$$z_n = \frac{i^n}{1+\sqrt{n}}, \quad n = 1, 2, \ldots .$$

Figure 1.5 suggests that (z_n) is a null sequence, and further evidence for this is provided by the inequality

$$|z_n| = \left|\frac{i^n}{1+\sqrt{n}}\right| = \frac{|i|^n}{1+\sqrt{n}} = \frac{1}{1+\sqrt{n}} \leq \frac{1}{\sqrt{n}}, \qquad (1.3)$$

which shows that $|z_n|$ is squeezed between 0 and $1/\sqrt{n}$. Since we know that the sequence $(1/\sqrt{n}\,)$ is null (by Exercise 1.2(a)), it seems likely that (z_n) must be null also. This is confirmed by the Squeeze Rule.

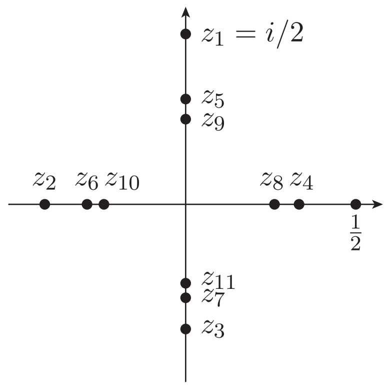


Figure 1.5 First eleven terms of the sequence $z_n = \dfrac{i^n}{1+\sqrt{n}}$, $n = 1, 2, \ldots$

Theorem 1.1 Squeeze Rule

If (a_n) is a real null sequence of non-negative terms, and if

$$|z_n| \leq a_n, \quad \text{for } n = 1, 2, \ldots,$$

then (z_n) is a null sequence.

Proof We need to show that for each positive number ε, there is an integer N such that

$$|z_n| < \varepsilon, \quad \text{for all } n > N. \qquad (1.4)$$

But, since (a_n) is null, there is an integer N such that

$$|a_n| < \varepsilon, \quad \text{for all } n > N.$$

We are given that $a_n \geq 0$ for all n, so $|a_n| = a_n$. Hence, with the value of N we have just found,

$$|z_n| \leq a_n < \varepsilon, \quad \text{for all } n > N,$$

so statement (1.4) does indeed hold. ■

When the inequality

$$|z_n| \leq a_n$$

holds for $n = 1, 2, \ldots$ (or even for all but a finite number of terms of the sequence), we say that the real sequence (a_n) **dominates** the sequence (z_n) (Figure 1.6). For example, we saw in inequality (1.3) that the sequence

$$z_n = i^n/(1+\sqrt{n}), \quad n = 1, 2, \ldots,$$

is dominated by the known null sequence

$$a_n = 1/\sqrt{n}, \quad n = 1, 2, \ldots;$$

thus (z_n) is null by the Squeeze Rule. This illustrates the *strategy* for using the Squeeze Rule – we suspect that (z_n) is null, and we prove it by choosing a suitable dominating null sequence. We ask you to apply this strategy in the following exercise.

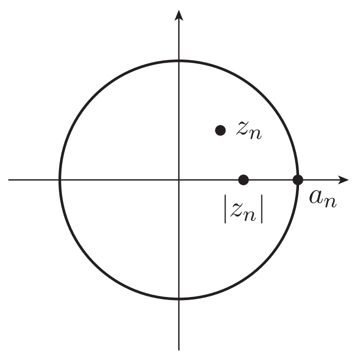


Figure 1.6 The terms z_n and a_n satisfy $|z_n| \leq a_n$

Exercise 1.3

(a) Prove that

$$z_n = \frac{(0.6 + 0.8i)^n}{n^2 + n}, \quad n = 1, 2, \ldots,$$

is a null sequence.

(*Hint*: First calculate $|0.6 + 0.8i|$.)

(b) Use the inequality $2^n \geq n$, for $n = 1, 2, \ldots$ (which can be proved by the Principle of Mathematical Induction), to prove that

$$z_n = (i/2)^n, \quad n = 1, 2, \ldots,$$

is a null sequence.

We now give two types of null sequences, namely $(1/n^p)$ for $p > 0$ and (α^n) for $|\alpha| < 1$, which we refer to as **basic null sequences**. The proof of the next theorem is postponed until Subsection 1.3.

Theorem 1.2 Basic Null Sequences

The following sequences are null:

(a) $\left(\dfrac{1}{n^p}\right)$, for $p > 0$

(b) (α^n), for $|\alpha| < 1$.

For example, choosing $p = \frac{1}{2}$ and $\alpha = 0.9i$ gives null sequences

$$\left(\frac{1}{\sqrt{n}}\right) \quad \text{and} \quad ((0.9i)^n),$$

both of which were discussed earlier.

Using the basic null sequences supplied by the theorem, it is possible to deduce the convergence of many sequences. For example, if

$$z_n = 1 + \left(\tfrac{1}{2}i\right)^n, \quad n = 1, 2, \ldots,$$

then

$$z_n - 1 = \left(\tfrac{1}{2}i\right)^n, \quad n = 1, 2, \ldots.$$

Since $\left(\left(\frac{1}{2}i\right)^n\right)$ is a basic null sequence (because $\left|\frac{1}{2}i\right| = \frac{1}{2} < 1$), the sequence (z_n) is convergent with limit $\alpha = 1$ (by Lemma 1.1). Usually, however, it is not so easy to recognise the limit of a sequence (even when this exists). Instead we can try to apply the following Combination Rules (which are proved in Subsection 1.3).

Theorem 1.3 Combination Rules for Sequences

If $\lim_{n\to\infty} z_n = \alpha$ and $\lim_{n\to\infty} w_n = \beta$, then

(a) **Sum Rule** $\lim_{n\to\infty}(z_n + w_n) = \alpha + \beta$

(b) **Multiple Rule** $\lim_{n\to\infty}(\lambda z_n) = \lambda\alpha$, where $\lambda \in \mathbb{C}$

(c) **Product Rule** $\lim_{n\to\infty}(z_n w_n) = \alpha\beta$

(d) **Quotient Rule** $\lim_{n\to\infty}\left(\frac{z_n}{w_n}\right) = \frac{\alpha}{\beta}$, provided that $\beta \neq 0$.

Remarks

1. When applying this theorem we usually refer to it simply as the 'Combination Rules', and similarly we refer to the 'Sum Rule', the 'Multiple Rule', and so forth. Later on we will meet other sorts of combination rules, so to avoid confusion we will sometimes refer to this theorem as the 'Combination Rules for sequences'.

   We follow similar conventions with other sets of rules of this type.

2. The Sum Rule can be described in words as 'the limit of the sum is the sum of the limits', and the other rules can be described in a similar way.

3. In applications of the Quotient Rule, it may happen that some of the terms w_n take the value 0, in which case z_n/w_n is not defined. We will see, however, in the proof of Theorem 1.3, that this can happen for only finitely many w_n (since $\beta \neq 0$), so w_n is *eventually* non-zero.

4. A special case of the Quotient Rule occurs when

   $$z_n = 1, \quad n = 1, 2, \ldots,$$

   and $\lim_{n\to\infty} w_n = \beta$, where $\beta \neq 0$. In this case, we deduce that

   $$\lim_{n\to\infty} \frac{1}{w_n} = \frac{1}{\beta}.$$

   This is sometimes referred to as the *Reciprocal Rule*; however, we reserve that label for a related result that appears later.

The following example illustrates how the Combination Rules are used, together with the basic null sequences from Theorem 1.2, to obtain the limits of more complicated sequences.

Example 1.2

Show that each of the following sequences is convergent, and find its limit.

(a) $z_n = \dfrac{i}{n} + \left(\dfrac{1+i}{2}\right)^n, \quad n = 1, 2, \ldots$

(b) $z_n = \dfrac{2in^2 + 3n + 2i}{3n^2 + in}, \quad n = 1, 2, \ldots$

(c) $z_n = \dfrac{6(1+i)^n + (3+2i)^n}{i(1+5i)^n + (1+i)^n}, \quad n = 1, 2, \ldots$

Solution

(a) We know that $(1/n)$ is a basic null sequence, and

$$\left(\frac{1+i}{2}\right)^n, \quad n = 1, 2, \ldots,$$

is also a basic null sequence because

$$\left|\frac{1+i}{2}\right| = \frac{\sqrt{2}}{2} < 1.$$

Hence, by the Sum and Multiple Rules,

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{i}{n} + \left(\frac{1+i}{2}\right)^n\right) &= i \lim_{n\to\infty}\frac{1}{n} + \lim_{n\to\infty}\left(\frac{1+i}{2}\right)^n \\ &= (i \times 0) + 0 = 0.\end{aligned}$$

(b) Although z_n is expressed as a quotient, we cannot apply the Quotient Rule immediately, because the sequences

$$2in^2 + 3n + 2i \quad \text{and} \quad 3n^2 + in, \quad n = 1, 2, \ldots,$$

do not appear to be convergent. Instead we rearrange the quotient in such a way that the Combination Rules can be applied. Dividing both the numerator and denominator by the *dominant term* n^2, that is, the highest power of n in the numerator and denominator, we obtain

$$z_n = \frac{2in^2 + 3n + 2i}{3n^2 + in} = \frac{2i + 3/n + 2i/n^2}{3 + i/n}.$$

Since $(1/n)$ and $(1/n^2)$ are basic null sequences, we find, by the Combination Rules, that

$$\lim_{n\to\infty} z_n = \frac{2i + 0 + 0}{3 + 0} = \frac{2i}{3}.$$

(c) Because $|1+i| = \sqrt{2}$, $|3+2i| = \sqrt{13}$ and $|1+5i| = \sqrt{26}$, the dominant term is $(1+5i)^n$, so we divide both the numerator and

the denominator of z_n by $(1+5i)^n$ to obtain

$$\begin{aligned} z_n &= \frac{6(1+i)^n + (3+2i)^n}{i(1+5i)^n + (1+i)^n} \\ &= \frac{6((1+i)/(1+5i))^n + ((3+2i)/(1+5i))^n}{i + ((1+i)/(1+5i))^n}. \end{aligned}$$

Since

$$|(1+i)/(1+5i)| = \sqrt{2}/\sqrt{26} < 1$$

and

$$|(3+2i)/(1+5i)| = \sqrt{13}/\sqrt{26} < 1,$$

we deduce that

$$\left(\frac{1+i}{1+5i}\right)^n \quad \text{and} \quad \left(\frac{3+2i}{1+5i}\right)^n, \quad n = 1, 2, \ldots,$$

are both basic null sequences. Hence, by the Combination Rules,

$$\lim_{n\to\infty} z_n = \frac{(6 \times 0) + 0}{i + 0} = 0.$$

Remark

Note the use of the convenient phrase 'by the Combination Rules' in Example 1.2(b) and (c) when several of these rules are being used.

Exercise 1.4

Show that each of the following sequences is convergent, and find its limit.

(a) $z_n = \dfrac{n^3 + 2in^2 + 3}{in^3 + 1}, \quad n = 1, 2, \ldots$

(b) $z_n = \dfrac{(3+i)^n + (2+2i)^n}{(1+2i)^n + 2(3+i)^n}, \quad n = 1, 2, \ldots$

We now present a theorem about convergent sequences that will be useful in Section 2.

Theorem 1.4

If $\lim\limits_{n\to\infty} z_n = \alpha$, then

(a) $\lim\limits_{n\to\infty} |z_n| = |\alpha|$

(b) $\lim\limits_{n\to\infty} \overline{z_n} = \overline{\alpha}$

(c) $\lim\limits_{n\to\infty} \operatorname{Re} z_n = \operatorname{Re} \alpha$

(d) $\lim\limits_{n\to\infty} \operatorname{Im} z_n = \operatorname{Im} \alpha.$

Figure 1.7 illustrates Theorem 1.4 for a given sequence (z_n).

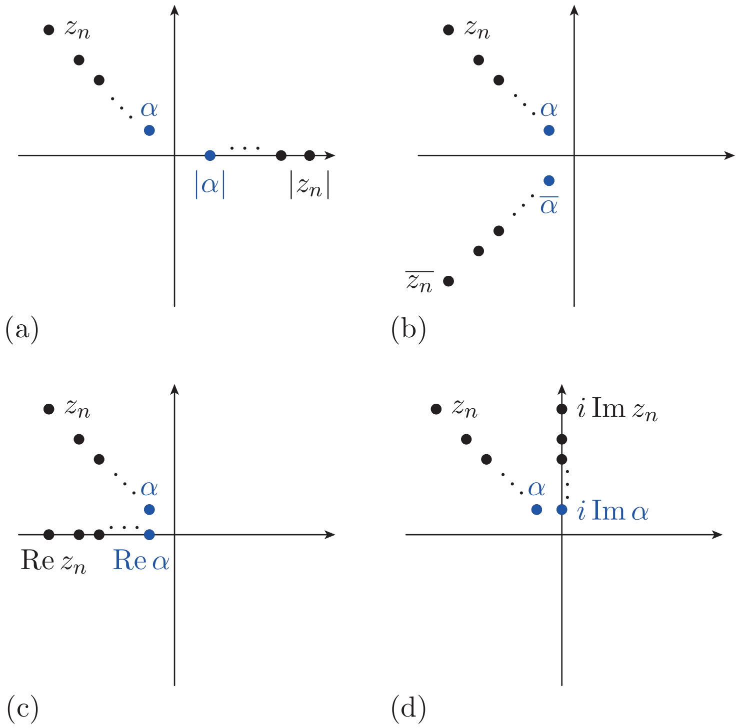


Figure 1.7 The convergent sequences of Theorem 1.4

Proof Let us prove part (a). Since $\lim_{n\to\infty} z_n = \alpha$, we know that $a_n = |z_n - \alpha|$, $n = 1, 2, \ldots$, is a null sequence with non-negative terms. By the backwards form of the Triangle Inequality (Theorem 5.1 of Unit A1),

$$||z_n| - |\alpha|| \leq |z_n - \alpha|, \quad \text{for } n = 1, 2, \ldots, \tag{1.5}$$

so the sequence $|z_n| - |\alpha|$, $n = 1, 2, \ldots$, is also null, by the Squeeze Rule. Hence $\lim_{n\to\infty} |z_n| = |\alpha|$, as required.

Parts (b), (c) and (d) can each be proved in a similar manner, but with inequality (1.5) replaced by one of the statements below (which themselves follow from Theorem 2.1(b) of Unit A1 and the inequalities $|\operatorname{Re} z| \leq |z|$ and $|\operatorname{Im} z| \leq |z|$).

For part (b), use $|\overline{z_n} - \overline{\alpha}| = |\overline{z_n - \alpha}| = |z_n - \alpha|$.

For part (c), use $|\operatorname{Re} z_n - \operatorname{Re}\alpha| = |\operatorname{Re}(z_n - \alpha)| \leq |z_n - \alpha|$.

For part (d), use $|\operatorname{Im} z_n - \operatorname{Im}\alpha| = |\operatorname{Im}(z_n - \alpha)| \leq |z_n - \alpha|$. ■

The next exercise asks you to prove a sort of converse to Theorem 1.4(c) and (d).

Exercise 1.5

Prove that if (x_n) and (y_n) are real sequences with $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} y_n = b$, then $\lim_{n\to\infty} (x_n + iy_n) = a + ib$.

Draw a diagram to illustrate your result.

1.2 Divergent sequences

Having commented earlier that some sequences appear not to be convergent, we now discuss such sequences in more detail.

Definition

A sequence that is not convergent is **divergent**.

Figure 1.8 (which is identical to Figure 1.1) shows two sequences, both of which appear to be divergent.

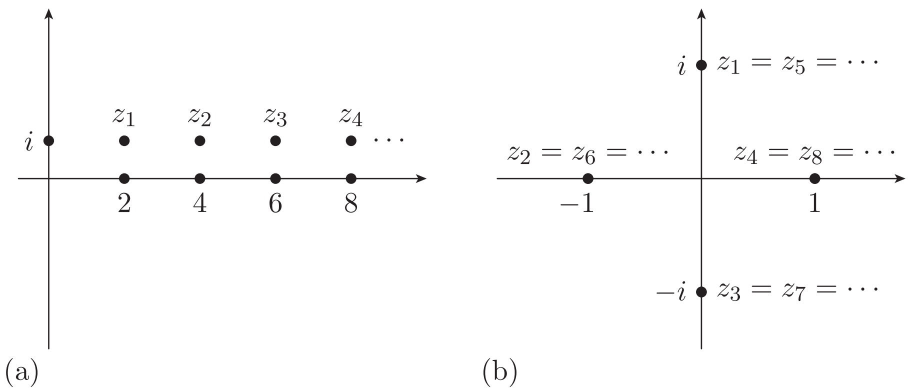


Figure 1.8 (a) $z_n = 2n + i,\ n = 1, 2, \ldots$ (b) $z_n = i^n,\ n = 1, 2, \ldots$

The sequence $(2n + i)$ is divergent because it does not settle down near any single point in $\mathbb{C}$. The sequence (i^n) makes a better attempt at converging, since the terms at least remain in the closed disc $\{z : |z| \leq 1\}$, but still they do not settle down near a *unique* finite limit.

It is often tricky to prove from the definition that a given sequence is divergent. Instead we will obtain two criteria that can be used to prove the divergence of a sequence. First, however, we need to discuss sequences that *tend to infinity*.

Definition

The sequence (z_n) **tends to infinity** if, for each positive number M, there is an integer N such that

$$|z_n| > M, \quad \text{for all } n > N.$$

In this case, we write

$$z_n \to \infty \text{ as } n \to \infty.$$

Observe that we do not usually write

$$\lim_{n\to\infty} z_n = \infty,$$

since this might suggest that ∞ is a complex number.

The geometric interpretation of this definition is that no matter how large a circle we consider, the terms z_n eventually lie *outside* this circle (Figure 1.9).

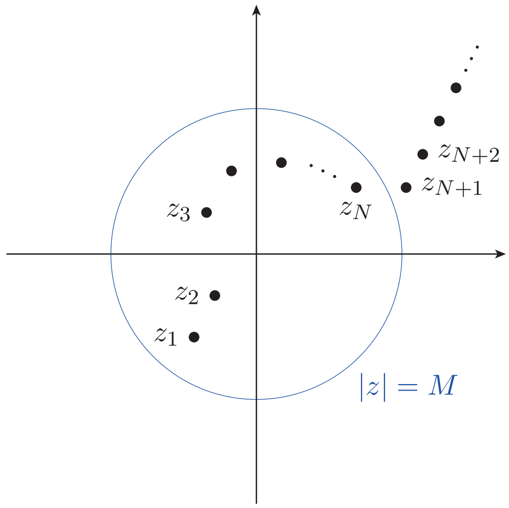


Figure 1.9 The sequence (z_n) eventually lies outside the circle $|z| = M$

For example, the sequence $(2n + i)$ tends to infinity because for each positive number M, there is an integer N such that

$$|2n + i| > M, \quad \text{for all } n > N.$$

In this case, we can choose N to be any positive integer greater than M because, with this choice,

$$|2n + i| = \sqrt{4n^2 + 1} > 2n > N > M, \quad \text{for all } n > N.$$

The following result enables us to use our knowledge of null sequences to identify sequences that tend to infinity.

Theorem 1.5 Reciprocal Rule for Sequences

Let (z_n) be a sequence. Then

$$z_n \to \infty \text{ as } n \to \infty$$

if and only if

$$(1/z_n) \text{ is a null sequence.}$$

In this theorem, the assumption that $z_n \to \infty$ as $n \to \infty$ implies that only a finite number of terms of (z_n) may be zero. We can therefore assume that these terms are omitted from (z_n) before forming $(1/z_n)$.

Proof First we prove that if $z_n \to \infty$ as $n \to \infty$, then $(1/z_n)$ is a null sequence. That is, we must show that for each positive number ε, there is an integer N such that

$$|1/z_n| < \varepsilon, \quad \text{for all } n > N. \tag{1.6}$$

But $z_n \to \infty$ as $n \to \infty$, so, by choosing $M = 1/\varepsilon$ in the definition of a sequence that tends to infinity, we can find an integer N such that

$$|z_n| > 1/\varepsilon, \quad \text{for all } n > N.$$

This statement is equivalent to statement (1.6), so $(1/z_n)$ is a null sequence.

The proof of the converse implication is similar. We must prove that if $(1/z_n)$ is a null sequence, then $z_n \to \infty$ as $n \to \infty$. That is, we must show that for each positive number M, there is an integer N such that

$$|z_n| > M, \quad \text{for all } n > N. \tag{1.7}$$

But the sequence $(1/z_n)$ is null, so, by choosing $\varepsilon = 1/M$ in the definition of null sequence, we can find an integer N such that

$$|1/z_n| < 1/M, \quad \text{for all } n > N.$$

This statement is equivalent to statement (1.7), so $z_n \to \infty$ as $n \to \infty$. ■

Remarks

1. The Reciprocal Rule does *not* assert that if (z_n) is null, then $1/z_n \to \infty$ as $n \to \infty$. This statement is false; for example, the sequence $z_n = 0$, $n = 1, 2, \ldots$, is null but $1/z_n$, $n = 1, 2, \ldots$, is not defined and the sequence $(1/z_n)$ does not exist.
2. For sequences with real terms, there is a distinction between tending to $+\infty$ and tending to $-\infty$ (Figure 1.10). No such distinction is possible with complex sequences, so there is a sense in which 'infinity' is a simpler place in complex analysis than in real analysis!

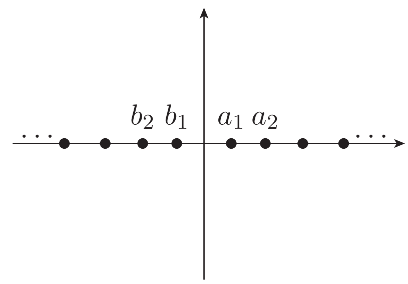


Figure 1.10 Real sequences with $a_n \to +\infty$ and $b_n \to -\infty$

Example 1.3

Use the Reciprocal Rule to prove that the following sequences tend to infinity.

(a) $z_n = in^2/2, \quad n = 1, 2, \ldots$

(b) $z_n = (3i)^n - 2^n, \quad n = 1, 2, \ldots$

Solution

(a) We have

$$\frac{1}{z_n} = \frac{2}{in^2} = \left(\frac{2}{i}\right)\frac{1}{n^2}, \quad \text{for } n = 1, 2, \ldots.$$

Since $(1/n^2)$ is a basic null sequence, we deduce that the sequence $(1/z_n)$ is null by the Multiple Rule. Hence the sequence (z_n) tends to infinity, by the Reciprocal Rule.

(b) We have

$$\frac{1}{z_n} = \frac{1}{(3i)^n - 2^n} = \frac{(1/(3i))^n}{1 - (2/(3i))^n}$$

(dividing the numerator and denominator by $(3i)^n$ because it is the dominant term).

Now $|1/(3i)| = 1/3 < 1$ and $|2/(3i)| = 2/3 < 1$, so

$$w_n = \left(\frac{1}{3i}\right)^n \quad \text{and} \quad w'_n = \left(\frac{2}{3i}\right)^n, \quad n = 1, 2, \ldots,$$

are basic null sequences. Hence, by the Combination Rules,

$$\lim_{n\to\infty}\left(\frac{1}{z_n}\right) = \frac{0}{1-0} = 0,$$

so the sequence (z_n) tends to infinity, by the Reciprocal Rule.

Exercise 1.6

Use the Reciprocal Rule to prove that the sequence

$$z_n = n^3 - in^2 + (1+i)n, \quad n = 1, 2, \ldots,$$

tends to infinity.

We now establish two criteria for divergence, both of which involve the idea of a *subsequence*. To explain this term, consider the sequence $z_n = (-1)^n$, $n = 1, 2, \ldots$. This splits naturally into two:

the even terms $z_2, z_4, z_6, \ldots, z_{2k}, \ldots,$ each of which equals 1;

the odd terms $z_1, z_3, z_5, \ldots, z_{2k-1}, \ldots,$ each of which equals -1.

Both of these are sequences in their own right, and we call them the **even subsequence** (z_{2k}) and the **odd subsequence** (z_{2k-1}) (where, in each case, the first term is given by $k = 1$).

In general, for a given sequence (z_n) we can consider many different subsequences, such as:

(z_{3k}), comprising the terms $z_3, z_6, z_9, \ldots$

(z_{4k+1}), comprising the terms $z_5, z_9, z_{13}, \ldots$

$(z_{k!})$, comprising the terms $z_1, z_2, z_6, \ldots$.

Definition

Let (n_k) be a sequence of positive integers that is strictly increasing; that is,

$$n_1 < n_2 < n_3 < \cdots.$$

Then the sequence (z_{n_k}) is a **subsequence** of the sequence (z_n).

Observe that any such sequence (n_k) satisfies

$$n_k \geq k, \quad \text{for } k = 1, 2, \ldots.$$

In the three examples above, $n_k = 3k$, $n_k = 4k+1$ and $n_k = k!$, respectively.

Exercise 1.7

Let $z_n = in/(n+1)$, $n = 1, 2, \ldots$. Write down the first four terms of each of the subsequences (z_{n_k}), where

(a) $n_k = 2k$ (b) $n_k = 4k - 1$ (c) $n_k = k^2$.

Now we can state our two criteria for establishing that a sequence is divergent. The proof of the following theorem is also postponed until Subsection 1.3.

Theorem 1.6 Subsequence Rules

(a) **First Subsequence Rule** The sequence (z_n) is divergent if (z_n) has two convergent subsequences with different limits.

(b) **Second Subsequence Rule** The sequence (z_n) is divergent if (z_n) has a subsequence that tends to infinity.

To see whether one of the Subsequence Rules can be applied to a given sequence, it is a good idea to write down the first few terms. For example,

$$z_n = (-1)^n i \quad \text{has terms } -i, i, -i, i, -i, i, \ldots,$$
$$w_n = n^{(-1)^n} \quad \text{has terms } 1, 2, \tfrac{1}{3}, 4, \tfrac{1}{5}, 6, \ldots,$$

for $n = 1, 2, \ldots$. The even subsequence of (z_n) has limit i, whereas the odd subsequence has limit $-i$, so (z_n) is divergent by the First Subsequence Rule. Also, the even subsequence of (w_n) tends to infinity, because $(2k)^{(-1)^{2k}} = 2k$, for $k = 1, 2, \ldots$, so (w_n) is divergent by the Second Subsequence Rule.

Exercise 1.8

Use the Subsequence Rules to prove that each of the following sequences is divergent.

(a) $z_n = i^n$, $n = 1, 2, \ldots$ (b) $z_n = n^2 \sin(n\pi/3)$, $n = 1, 2, \ldots$

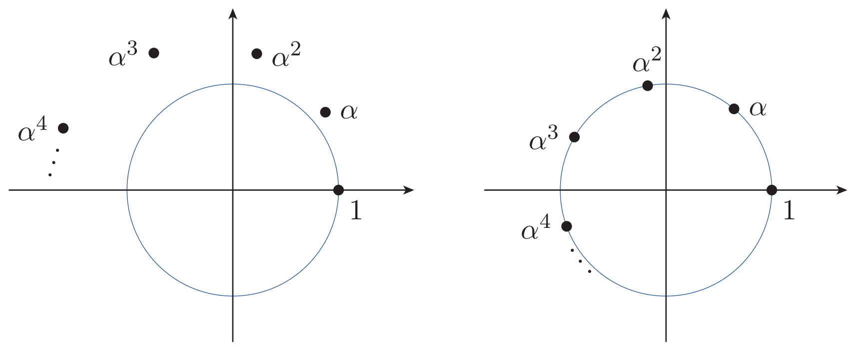
In Theorem 1.2(b) you saw that the sequence (α^n) is null if $|\alpha| < 1$, and it is clear that (α^n) is convergent if $\alpha = 1$. We end this subsection by proving that (α^n) is divergent for all other values of α.

Theorem 1.7

(a) If $|\alpha| > 1$, then the sequence (α^n) tends to infinity.

(b) If $|\alpha| = 1$ and $\alpha \neq 1$, then the sequence (α^n) is divergent.

Figure 1.11 illustrates Theorem 1.7 for two given sequences (α^n).



(a) (b)

Figure 1.11 The sequence (α^n) with (a) $|\alpha| > 1$, (b) $|\alpha| = 1$, $\alpha \neq 1$

Proof

(a) By Theorem 1.2(b), if $|\alpha| > 1$ and $\beta = 1/\alpha$, then (β^n) is a basic null sequence, because $|\beta| = |1/\alpha| < 1$. Therefore the sequence (α^n) tends to infinity by the Reciprocal Rule.

(b) We use proof by contradiction. Assume that the sequence (α^n), where $|\alpha| = 1$ and $\alpha \neq 1$, converges, to some number $\beta \in \mathbb{C}$, say. Then

$$\lim_{n\to\infty} \alpha^{n+1} = \alpha \lim_{n\to\infty} \alpha^n = \alpha\beta \quad \text{(Multiple Rule)},$$

and also

$$\lim_{n\to\infty} \alpha^{n+1} = \beta,$$

since the sequence (α^{n+1}) is identical to the sequence (α^n), except that the first term is missing. Hence

$$\alpha\beta = \beta,$$

so $\beta = 0$, since $\alpha \neq 1$ (by hypothesis). We have deduced that (α^n) is a null sequence, which is impossible, because $|\alpha^n| = |\alpha|^n = 1$, for $n = 1, 2, \ldots$. Thus we reject the initial assumption and conclude that the sequence (α^n) is divergent. ■

1.3 Proofs of sequence theorems

Here we give proofs of the theorem on basic null sequences, the Combination Rules and the Subsequence Rules. This subsection may be omitted on a first reading.

Theorem 1.2 Basic Null Sequences

The following sequences are null:

(a) $\left(\dfrac{1}{n^p}\right)$, for $p > 0$

(b) (α^n), for $|\alpha| < 1$.

Proof

(a) We need to show that for each positive number ε, there is an integer N such that

$$\frac{1}{n^p} < \varepsilon, \quad \text{for all } n > N. \tag{1.8}$$

Since $p > 0$,

$$\frac{1}{n^p} < \varepsilon \iff n^p > \frac{1}{\varepsilon} \iff n > (1/\varepsilon)^{1/p}.$$

Hence statement (1.8) holds if we take N to be any positive integer greater than $(1/\varepsilon)^{1/p}$. Thus $(1/n^p)$ is a null sequence.

(b) If $\alpha = 0$, then the result is evident. If $0 < |\alpha| < 1$, then we can define $a = (1/|\alpha|) - 1 > 0$, so

$$|\alpha| = \frac{1}{1+a}.$$

Now, by the Binomial Theorem,

$$(1+a)^n = 1 + na + \frac{n(n-1)}{2!}a^2 + \cdots + a^n \geq na,$$

for $n = 1, 2, \ldots$, since $a > 0$. Hence

$$|\alpha|^n = \frac{1}{(1+a)^n} \leq \frac{1}{na} = \frac{(1/a)}{n}, \quad \text{for } n = 1, 2, \ldots.$$

We know that $(1/n)$ is a null sequence by part (a), so

$$a_n = \frac{(1/a)}{n}, \quad n = 1, 2, \ldots,$$

is also a null sequence, by the Multiple Rule (to be proved shortly). Hence (α^n) is a null sequence, by the Squeeze Rule. ■

In order to prove the Combination Rules, we need the following lemma (which will be used in the proof of the Product Rule). It is also needed later to prove the Second Subsequence Rule.

Lemma 1.2

If (z_n) is a convergent sequence, then there is a positive number M such that

$$|z_n| \leq M, \quad \text{for } n = 1, 2, \ldots. \tag{1.9}$$

Proof Suppose that $\lim_{n\to\infty} z_n = \alpha$. Then, by taking $\varepsilon = 1$ in the definition of convergence, we see that there is an integer N such that

$$|z_n - \alpha| < 1, \quad \text{for all } n > N.$$

Now, by the Triangle Inequality,

$$|z_n| = |(z_n - \alpha) + \alpha| \leq |z_n - \alpha| + |\alpha|,$$

so

$$|z_n| < 1 + |\alpha|, \quad \text{for all } n > N.$$

Thus if $M = \max\{|z_1|, |z_2|, \ldots, |z_N|, 1 + |\alpha|\}$, then

$$|z_n| \leq M, \quad \text{for } n = 1, 2, \ldots .$$

■

If a sequence (z_n) satisfies statement (1.9) for a positive number M, then all the terms of the sequence lie in a closed disc with centre 0 and radius M. In this case, we say that the sequence is **bounded**. Thus the lemma states that 'every convergent sequence is bounded'.

We are now in a position to prove the Combination Rules.

Theorem 1.3 Combination Rules for Sequences

If $\lim\limits_{n\to\infty} z_n = \alpha$ and $\lim\limits_{n\to\infty} w_n = \beta$, then

(a) **Sum Rule** $\lim\limits_{n\to\infty}(z_n + w_n) = \alpha + \beta$

(b) **Multiple Rule** $\lim\limits_{n\to\infty}(\lambda z_n) = \lambda\alpha, \quad$ where $\lambda \in \mathbb{C}$

(c) **Product Rule** $\lim\limits_{n\to\infty}(z_n w_n) = \alpha\beta$

(d) **Quotient Rule** $\lim\limits_{n\to\infty}\left(\dfrac{z_n}{w_n}\right) = \dfrac{\alpha}{\beta}, \quad$ provided that $\beta \neq 0$.

Proof

(a) We need to show that for each positive number ε, there is an integer N such that

$$|(z_n + w_n) - (\alpha + \beta)| < \varepsilon, \quad \text{for all } n > N. \tag{1.10}$$

Now

$$\begin{aligned}|(z_n + w_n) - (\alpha + \beta)| &= |(z_n - \alpha) + (w_n - \beta)| \\ &\leq |z_n - \alpha| + |w_n - \beta|,\end{aligned}$$

by the Triangle Inequality. Since $\lim\limits_{n\to\infty} z_n = \alpha$ and $\lim\limits_{n\to\infty} w_n = \beta$, we know that there are integers N_1, N_2 such that

$$\begin{aligned}|z_n - \alpha| &< \tfrac{1}{2}\varepsilon, \quad \text{for all } n > N_1, \\ |w_n - \beta| &< \tfrac{1}{2}\varepsilon, \quad \text{for all } n > N_2.\end{aligned}$$

(We use $\frac{1}{2}\varepsilon$ here, in order to obtain ε in (1.10).) So if we choose N to be any integer greater than both N_1 and N_2, then, for all $n > N$,

$$|(z_n + w_n) - (\alpha + \beta)| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon,$$

as required.

(b) We need to show that for each positive number ε, there is an integer N such that

$$|\lambda z_n - \lambda\alpha| < \varepsilon, \quad \text{for all } n > N. \tag{1.11}$$

If $\lambda = 0$, then this inequality is evident, so we assume that $\lambda \neq 0$. Since $\lim\limits_{n\to\infty} z_n = \alpha$, we know that there is an integer N such that

$$|z_n - \alpha| < \frac{\varepsilon}{|\lambda|}, \quad \text{for all } n > N.$$

(We use $\varepsilon/|\lambda|$ here in order to obtain ε in (1.11).) Hence

$$|\lambda z_n - \lambda\alpha| = |\lambda||z_n - \alpha| < \varepsilon, \quad \text{for all } n > N,$$

as required.

(c) The idea here is to express the difference $z_n w_n - \alpha\beta$ in terms of $z_n - \alpha$ and $w_n - \beta$, as follows:

$$z_n w_n - \alpha\beta = z_n(w_n - \beta) + \beta(z_n - \alpha).$$

By the Multiple Rule,

$$\lim_{n\to\infty} \beta(z_n - \alpha) = \beta \lim_{n\to\infty} (z_n - \alpha) = 0.$$

Also, by Lemma 1.2, there is a positive number M such that

$$|z_n| \leq M, \quad \text{for } n = 1, 2, \ldots;$$

hence

$$|z_n(w_n - \beta)| = |z_n||w_n - \beta| \leq M|w_n - \beta|, \quad \text{for } n = 1, 2, \ldots. \tag{1.12}$$

The Multiple Rule tells us that $\lim\limits_{n\to\infty} M|w_n - \beta| = 0$, so, by applying the Squeeze Rule, we can see from inequality (1.12) that

$$\lim_{n\to\infty} z_n(w_n - \beta) = 0.$$

Thus, by the Sum Rule and Multiple Rule,

$$\begin{aligned}\lim_{n\to\infty} (z_n w_n - \alpha\beta) &= \lim_{n\to\infty} z_n(w_n - \beta) + \lim_{n\to\infty} \beta(z_n - \alpha)\\ &= 0 + 0 = 0,\end{aligned}$$

as required.

(d) Once again, we write the required difference in terms of $z_n - \alpha$ and $w_n - \beta$, as follows:

$$\frac{z_n}{w_n} - \frac{\alpha}{\beta} = \frac{\beta(z_n - \alpha) - \alpha(w_n - \beta)}{w_n\beta}.$$

We know that the numerator sequence $(\beta(z_n - \alpha) - \alpha(w_n - \beta))$ is null by the Sum and Multiple Rules, but there is a problem with the denominator. Some of the terms w_n may be 0, in which case z_n/w_n is undefined. However, we will show that eventually $|w_n|$ is positive; in fact, we will show that there is an integer N such that

$$|w_n| > \tfrac{1}{2}|\beta|, \quad \text{for all } n > N \tag{1.13}$$

(see Figure 1.12).

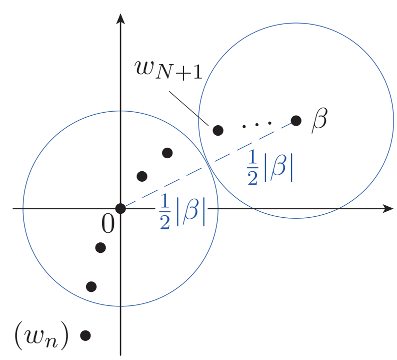


Figure 1.12 Eventually $|w_n| > \frac{1}{2}|\beta|$

To prove statement (1.13), we take $\varepsilon = \frac{1}{2}|\beta|$ in the definition of convergence, and choose an integer N such that

$$|w_n - \beta| < \tfrac{1}{2}|\beta|, \quad \text{for all } n > N.$$

Then, for all $n > N$,

$$\tfrac{1}{2}|\beta| > |w_n - \beta| \geq |\beta| - |w_n| \implies |w_n| > |\beta| - \tfrac{1}{2}|\beta| = \tfrac{1}{2}|\beta|$$

by the backwards form of the Triangle Inequality. Thus, for all $n > N$,

$$\begin{aligned}\left|\frac{z_n}{w_n} - \frac{\alpha}{\beta}\right| &= \frac{|\beta(z_n - \alpha) - \alpha(w_n - \beta)|}{|w_n||\beta|} \\ &< \frac{2}{|\beta|^2}|\beta(z_n - \alpha) - \alpha(w_n - \beta)| \quad \text{(by (1.13))}.\end{aligned}$$

Since the right-hand side defines a real null sequence of non-negative terms (by the Combination Rules), we deduce that $\left(\dfrac{z_n}{w_n} - \dfrac{\alpha}{\beta}\right)$ is null, by the Squeeze Rule. ■

Theorem 1.6 Subsequence Rules

(a) **First Subsequence Rule** The sequence (z_n) is divergent if (z_n) has two convergent subsequences with different limits.

(b) **Second Subsequence Rule** The sequence (z_n) is divergent if (z_n) has a subsequence that tends to infinity.

Proof

(a) Since a given sequence is either convergent or divergent, it is sufficient to prove that if (z_n) is convergent with limit α, then each subsequence (z_{n_k}) is also convergent with limit α. We need to show that for each positive number ε, there is an integer K such that

$$|z_{n_k} - \alpha| < \varepsilon, \quad \text{for all } k > K.$$

Since $\lim\limits_{n\to\infty} z_n = \alpha$, we know that there is an integer N such that

$$|z_n - \alpha| < \varepsilon, \quad \text{for all } n > N.$$

Let us choose $K = N$. Then $n_K \geq K = N$, so

$$n_k > n_K \geq N, \quad \text{for all } k > K,$$

and hence

$$|z_{n_k} - \alpha| < \varepsilon, \quad \text{for all } k > K.$$

(b) If (z_n) has a subsequence that tends to infinity, then (z_n) cannot be bounded. Hence (z_n) cannot be convergent, by Lemma 1.2. ■

Further exercises

Exercise 1.9

Prove that the following sequence is null,

$$z_n = (1+i)/(2n^2 - 1), \; n = 1, 2, \ldots,$$

(a) by using the definition (b) by using the Squeeze Rule.

Exercise 1.10

Decide which of the following sequences are null, and justify your answers.

(a) $z_n = \left(\frac{1}{2} + \frac{i}{2}\right)^n, \quad n = 1, 2, \ldots$ (b) $z_n = \frac{1}{2} + \left(\frac{i}{2}\right)^n, \quad n = 1, 2, \ldots$

(c) $z_n = (1+i)^n, \quad n = 1, 2, \ldots$

Exercise 1.11

Show that each of the following sequences is convergent, and find its limit.

(a) $z_n = 5 + \frac{i}{2n}, \quad n = 1, 2, \ldots$ (b) $z_n = \frac{2n - i}{n^2}, \quad n = 1, 2, \ldots$

(c) $z_n = \frac{n - i}{n + i}, \quad n = 1, 2, \ldots$ (d) $z_n = \frac{n^3 + 3in - 2}{4n^3 - in^2}, \quad n = 1, 2, \ldots$

(e) $z_n = \frac{(1+i)^n + (\sqrt{3} - i)^n}{3(2 - 2i)^n - 1}, \quad n = 1, 2, \ldots$

Exercise 1.12

Decide which of the following sequences tend to infinity, and justify your answers.

(a) $z_n = \frac{n}{i}, \quad n = 1, 2, \ldots$ (b) $z_n = e^{in}, \quad n = 1, 2, \ldots$

(c) $z_n = \frac{(\sqrt{3} - i)^n - 1}{(1+i)^n}, \quad n = 1, 2, \ldots$

Exercise 1.13

Prove that each of the following sequences is divergent.

(a) $z_n = (i-1)^n, \quad n = 1, 2, \ldots$ (b) $z_n = e^{n\pi i}, \quad n = 1, 2, \ldots$

(c) $z_n = n\cos(n\pi i^n), \quad n = 1, 2, \ldots$

Exercise 1.14

Suppose that the sequence (z_n) can be separated into two subsequences (z_{m_k}) and (z_{n_k}) that both converge to α.

Prove that

$$\lim_{n \to \infty} z_n = \alpha.$$

2 Continuous functions

After working through this section, you should be able to:

- explain the statements 'the function f is continuous at the point α' and 'the function f is discontinuous at the point α'
- show that a function is *continuous/discontinuous* at a point by working from the definitions
- use the Combination Rules, the Composition Rule and the Restriction Rule for continuous functions
- recognise certain *basic continuous functions*
- use the continuity of functions to evaluate the limits of certain sequences.

2.1 Continuity: sequential definition

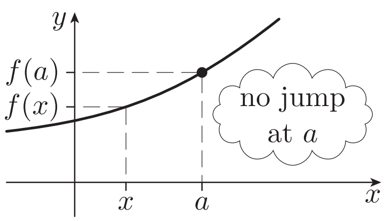


Figure 2.1 f continuous at a

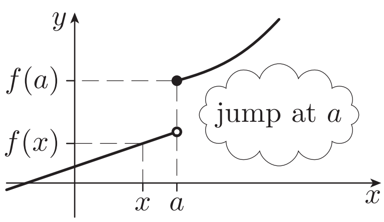


Figure 2.2 f not continuous at a

The word *continuity* means the property of being continuous. In real analysis, a continuous function is, loosely speaking, a real function that does not have jumps in its graph (compare Figures 2.1 and 2.2). Speaking informally still, we say that the real function f is continuous at a point a provided that

if x tends to a, then $f(x)$ tends to $f(a)$.

In this module, we will take for granted that the following real functions are continuous at each point of their (real) domains:

polynomial functions, rational functions,
trigonometric and exponential functions and their inverses.

The continuity of a function f at a point α is important also in complex analysis and, by analogy, it can be described roughly as follows:

if z tends to α, then $f(z)$ tends to $f(\alpha)$.

In this subsection we make this informal idea precise, using an approach based on the convergence of sequences.

Because we deal with so many convergent sequences, we will often write simply '$z_n \to \alpha$', omitting 'as $n \to \infty$'.

Later, in Subsection 2.2, we introduce an equivalent definition of continuity, the so-called ε-δ definition. Like ε, the Greek lower-case letter δ (delta) is also commonly used to denote a positive number in real and complex analysis.

Try the following exercise; its result will be used to help explain the sequential definition of continuity.

Exercise 2.1

Let (z_n) be a sequence such that $\lim_{n\to\infty} z_n = i$. Determine

$$\lim_{n\to\infty} (z_n^2 + 3z_n).$$

You should have found from Exercise 2.1 that if (z_n) is a sequence that tends to i, then the sequence $(z_n^2 + 3z_n)$ tends to $-1+3i$. That is,

$$z_n \to i \implies z_n^2 + 3z_n \to -1+3i.$$

By introducing the function $f(z) = z^2 + 3z$, we can rewrite this statement as

$$z_n \to i \implies f(z_n) \to f(i).$$

You should read this as 'z_n tends to i implies $f(z_n)$ tends to $f(i)$'. In these circumstances we say that f is *continuous* at the point i.

Figure 2.3 illustrates the fact that this function f is continuous at i. On the left of the figure there is an arbitrary sequence (z_n) converging to i. On the right is the sequence obtained by applying f to each point z_n; the resulting sequence $(f(z_n))$ converges to $-1+3i$.

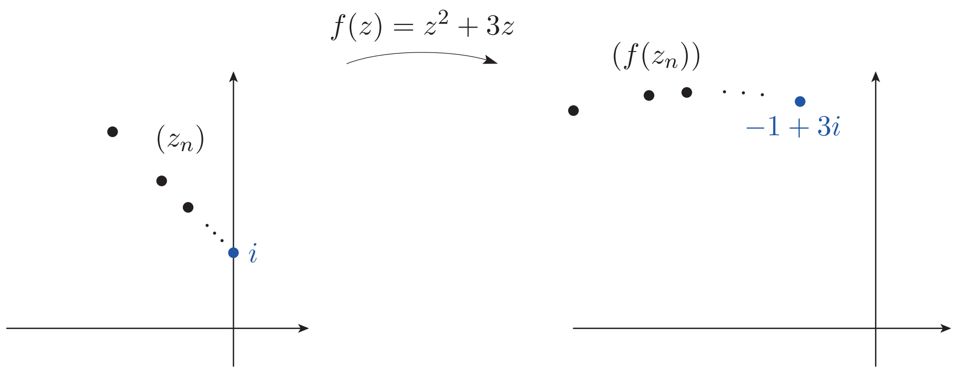


Figure 2.3 Image of a convergent sequence under $f(z) = z^2 + 3z$

Let us now give a general definition of what it means to be continuous, in terms of sequences.

Definition

Let $f\colon A \longrightarrow \mathbb{C}$ and $\alpha \in A$. Then f is **continuous at $\boldsymbol{\alpha}$** if, for each sequence (z_n) in A such that $z_n \to \alpha$, we have

$$f(z_n) \to f(\alpha);$$

that is,

$$z_n \to \alpha \implies f(z_n) \to f(\alpha).$$

If f is continuous at each α in A, then we say that f is **continuous** (on A).

The domain A is often the whole complex plane $\mathbb{C}$, or it may be a subset of $\mathbb{C}$ such as $\mathbb{C} - \{0\}$ or the real line $\mathbb{R}$.

The next example shows you how to prove that a function is continuous.

Example 2.1

Prove that the function $f(z) = 1/z$ is continuous.

Solution

The domain of f is $\mathbb{C} - \{0\}$ because f is defined everywhere except at the point 0.

Take $\alpha \in \mathbb{C} - \{0\}$. If (z_n) is a sequence in $\mathbb{C} - \{0\}$ such that $z_n \to \alpha$, then $1/z_n \to 1/\alpha$, by the Quotient Rule. That is, $f(z_n) \to f(\alpha)$. Therefore f is continuous at α for every point α in $\mathbb{C} - \{0\}$, so f is continuous (on $\mathbb{C} - \{0\}$).

You may find Theorem 1.4 helpful in attempting the following exercise.

Exercise 2.2

Prove that the following functions are continuous.

(a) $f(z) = 1$ (b) $f(z) = z$ (c) $f(z) = \overline{z}$ (d) $f(z) = \operatorname{Re} z$

(e) $f(z) = \operatorname{Im} z$ (f) $f(z) = |z|$

So far in this subsection we have looked at the definition of a continuous function, but of course not all functions are continuous. Here is an exercise that illustrates this.

Exercise 2.3

Let $z_n = e^{i(\pi + 1/n)}$, $n = 1, 2, \ldots$. Determine

$$\lim_{n\to\infty} z_n \quad \text{and} \quad \lim_{n\to\infty} \operatorname{Arg} z_n.$$

Also, write down $\operatorname{Arg}(-1)$.

(*Hint*: Plotting a few terms of (z_n) should help you.)

You should have found that the sequence (z_n) tends to -1, but the sequence $(\operatorname{Arg} z_n)$ does *not* tend to $\operatorname{Arg}(-1) = \pi$. If we use the symbol $\not\to$ to mean 'does not tend to', then we can express this observation as

$$z_n \to -1 \quad \text{but} \quad \operatorname{Arg} z_n \not\to \pi.$$

We now introduce the function $f(z) = \operatorname{Arg} z$ (with domain $\mathbb{C} - \{0\}$), so the statement above can be rewritten as

$$z_n \to -1 \quad \text{but} \quad f(z_n) \not\to f(-1).$$

This statement tells us that the function $f(z) = \operatorname{Arg} z$ is not continuous at the point -1. We say that f is *discontinuous* at -1. The situation is illustrated in Figure 2.4.

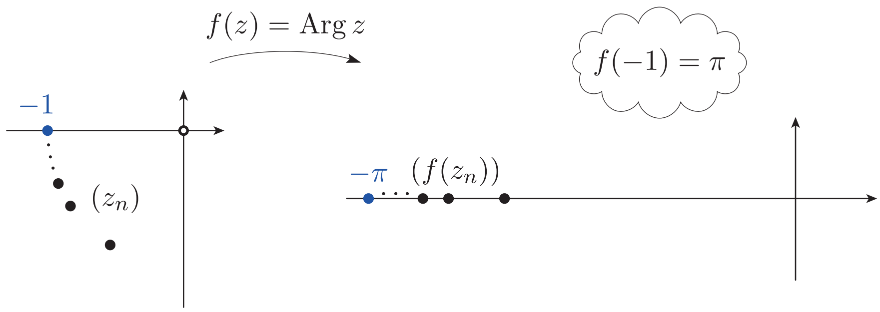


Figure 2.4 Image of a sequence that converges to -1 under $f(z) = \operatorname{Arg} z$

The general definition of what it means to be discontinuous is as follows.

Definition

Let $f : A \longrightarrow \mathbb{C}$ and $\alpha \in A$. If f is not continuous at α, then we say that f is **discontinuous at** $\boldsymbol{\alpha}$.

The strategy below can be used to decide where a function is continuous and where it is discontinuous.

Strategy for determining whether a function is continuous

To determine whether a function $f : A \longrightarrow \mathbb{C}$ is continuous at a point α in A, apply the following steps.

1. Guess whether f is continuous or discontinuous at α.
2. If you believe that f is continuous at α, then check that
$$z_n \to \alpha \implies f(z_n) \to f(\alpha)$$
for *every* sequence (z_n) in A that tends to α.
3. If you believe that f is discontinuous at α, then find *just one* sequence (z_n) in A such that $z_n \to \alpha$ but $f(z_n) \not\to f(\alpha)$.

You have already practised proving that functions are continuous, in Exercise 2.2. The next exercise gives you practice at proving that a function is discontinuous at certain points.

Exercise 2.4

Prove that

$$f(z) = \operatorname{Arg} z$$

is discontinuous at all $\alpha \in \mathbb{R}$ with $\alpha < 0$.

For an elaborate function such as

$$r(z) = \frac{1 - 3iz}{z^2 - 4}$$

it can be tricky to check continuity directly from the definition. Fortunately, as with limits of sequences, we can save ourselves some work by using the Combination Rules. We delay consideration of the proof of the following result until Subsection 2.3.

Theorem 2.1 Combination Rules for Continuous Functions

If f and g are functions that are continuous at α, then

(a) **Sum Rule** $f + g$ is continuous at α

(b) **Multiple Rule** λf is continuous at α, for $\lambda \in \mathbb{C}$

(c) **Product Rule** fg is continuous at α

(d) **Quotient Rule** f/g is continuous at α, provided that $g(\alpha) \neq 0$.

Starting with the functions $f(z) = 1$ and $f(z) = z$, which are both continuous, by Exercise 2.2, we can prove that the function r above is continuous at all points other than 2 and -2 by applying the Combination Rules. Since the domain of r is $\mathbb{C} - \{2, -2\}$, we deduce that r is a continuous function.

Reasoning in a similar way, we see that any polynomial function $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ is continuous, and so is any rational function $r(z) = p(z)/q(z)$ (where p and q are polynomial functions). Here you should remember that when we say that r is continuous, we mean that it is continuous on its domain, which is the set of points z for which $q(z)$ is not zero.

The Combination Rules do not tell us that a function such as $h(z) = |z^2 + 1|$ is continuous. There is, however, a helpful rule about compositions of continuous functions which can be used instead. We state this rule now, and again postpone its proof until Subsection 2.3.

Theorem 2.2 Composition Rule for Continuous Functions

Let f be a function that is continuous at α, and let g be a function that is continuous at $f(\alpha)$. Then $g \circ f$ is continuous at α.

To apply this rule to the function $h(z) = |z^2 + 1|$, we define $f(z) = z^2 + 1$ and $g(z) = |z|$, so

$$(g \circ f)(z) = g(f(z)) = |z^2 + 1|.$$

That is, $g \circ f = h$. The function f is continuous because it is a polynomial function, and you were asked to show that the function g is continuous in Exercise 2.2(f). We deduce from the Composition Rule that h is continuous on its domain $\mathbb{C}$.

Here is another example that involves both the Combination and Composition Rules.

Example 2.2

Prove that the function $f(z) = e^z$ is continuous.

Solution

By definition, $e^z = e^x(\cos y + i \sin y)$, where $z = x + iy$. Since $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$, we see that

$$e^z = e^{\operatorname{Re} z}(\cos(\operatorname{Im} z) + i \sin(\operatorname{Im} z)).$$

We saw in Exercise 2.2 that the real-valued functions $f(z) = \operatorname{Re} z$ and $f(z) = \operatorname{Im} z$ are both continuous. Furthermore, we know that, as stated at the start of this subsection, the *real* trigonometric and exponential functions are also continuous. We can now apply the Composition Rule and then the Combination Rules to see that $f(z) = e^z$ is continuous on $\mathbb{C}$.

Using methods similar to those of the preceding example, we can show that the complex trigonometric and hyperbolic functions are continuous.

Another useful technique for proving that a function is continuous is to check whether it is the restriction of some other function that we already know to be continuous. This rule is stated below and proved in Subsection 2.3.

Theorem 2.3 Restriction Rule for Continuous Functions

Let f and g be complex functions with domains A and B, respectively, and let $A \subseteq B$. If

- $f(z) = g(z)$, for $z \in A$
- g is continuous at $\alpha \in A$,

then f is continuous at α.

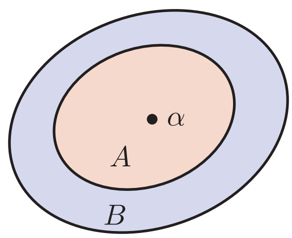


Figure 2.5 A point α in a set A contained in a set B

The domains A and B of f and g are illustrated in Figure 2.5.

For an example of the Restriction Rule, the function $\gamma(t) = t + it^2$ with domain $A = \mathbb{R}$ is continuous, because it is the restriction to $A = \mathbb{R}$ of the polynomial function $g(z) = z + iz^2$ with domain $B = \mathbb{C}$.

This function γ is a parametrisation of the path Γ shown on the right of Figure 2.6, which is a parabola traversed from left to right. The Restriction Rule is particularly useful for proving that parametrisations of paths are continuous.

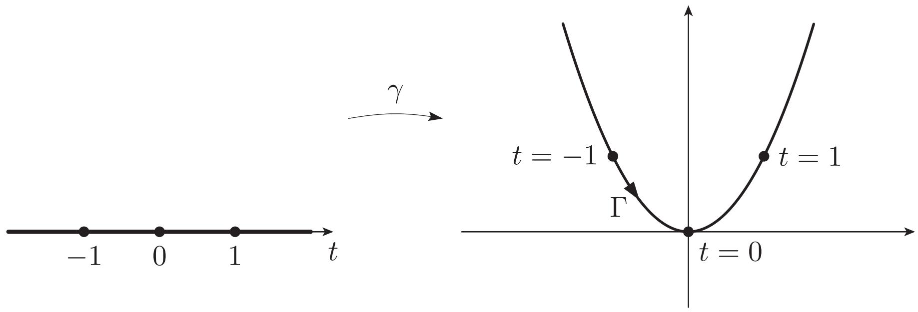


Figure 2.6 Image of the real line under $\gamma(t) = t + it^2$

Try the next exercise, in which you will use all the rules you have met for proving that functions are continuous.

Exercise 2.5

Use the rules above to determine whether the following functions are continuous.

(a) $f(z) = e^{-z^2}$ (b) $f(x) = \dfrac{x^2+i}{x^2-i}$ $(x \in \mathbb{R})$ (c) $f(z) = \log|z|$

(d) $f(z) = \operatorname{Re}(z^2+1) - |z|^2$

2.2 Continuity: ε-δ definition

In this subsection we present another definition of continuity, which is equivalent to the sequential definition of continuity that we met in the previous subsection. To motivate this new definition, consider the continuous function $f(z) = 2iz$. The geometric effect of this function is to scale points by a factor of 2 and rotate them by $\pi/2$ anticlockwise about the origin. If (z_n) is a sequence that converges to a point α, then the sequence $(f(z_n))$ converges to $f(\alpha) = 2i\alpha$, as shown in Figure 2.7.

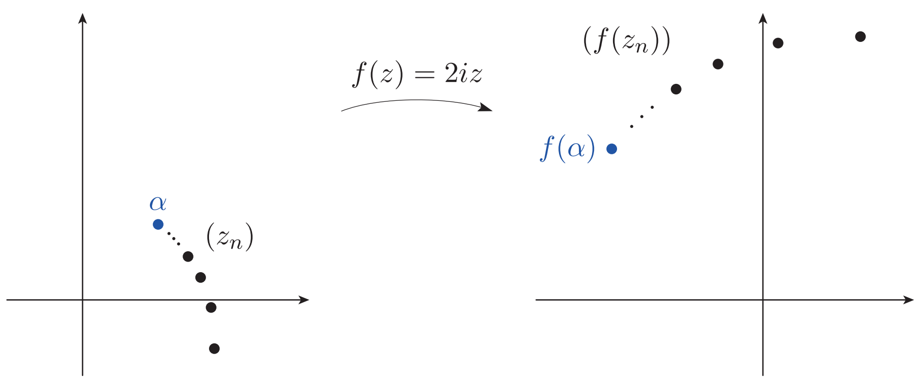


Figure 2.7 Image of a convergent sequence under $f(z) = 2iz$

Now look at Figure 2.8. On the right, there is a disc of radius ε centred at $f(\alpha)$, and on the left, there is a disc of radius $\frac{1}{2}\varepsilon$ centred at α. The disc on the left comprises those points z that satisfy $|z-\alpha| < \frac{1}{2}\varepsilon$.

If z belongs to this disc, then

$$|f(z) - f(\alpha)| = |2iz - 2i\alpha| = 2|z - \alpha| < 2 \times \tfrac{1}{2}\varepsilon = \varepsilon,$$

so $f(z)$ belongs to the disc on the right.

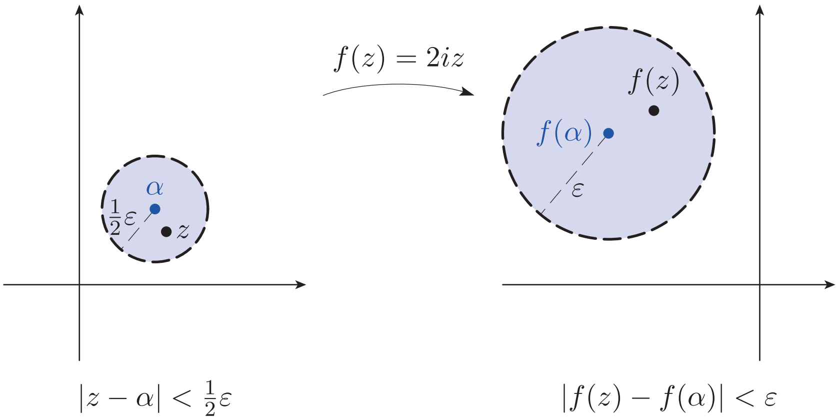


Figure 2.8 Image of a disc centred at α under $f(z) = 2iz$

Another way of expressing the statement

$$|z - \alpha| < \tfrac{1}{2}\varepsilon \implies |f(z) - f(\alpha)| < \varepsilon$$

is to say that

$$|f(z) - f(\alpha)| < \varepsilon, \quad \text{for all } z \in \mathbb{C} \text{ with } |z - \alpha| < \tfrac{1}{2}\varepsilon.$$

This motivates the second definition of continuity.

Definition

Let $f\colon A \longrightarrow \mathbb{C}$ and $\alpha \in A$. Then f is **continuous at $\boldsymbol{\alpha}$** if, for each $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|f(z) - f(\alpha)| < \varepsilon, \quad \text{for all } z \in A \text{ with } |z - \alpha| < \delta.$$

Put another way, f is continuous at α if, given any disc of radius ε centred at $f(\alpha)$, we can find a disc of radius δ centred at α small enough that its image under f lies within the disc of radius ε. This situation is illustrated in Figure 2.9.

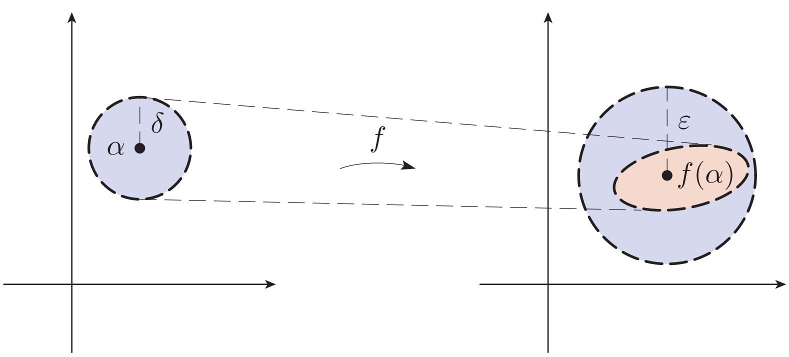


Figure 2.9 The image of the disc of radius δ centred at α is contained in the disc of radius ε centred at $f(\alpha)$

Usually δ depends on ε. For example, in the case of the map $f(z) = 2iz$, we found that we could take $\delta = \frac{1}{2}\varepsilon$, and in fact we can take δ to be any positive number smaller than $\frac{1}{2}\varepsilon$.

To demonstrate how to apply the ε-δ definition of continuity, consider the function $f(z) = \operatorname{Arg} z$. We saw in Exercise 2.4 that f is discontinuous at points on the negative real axis, and it is not defined at $z = 0$. The next example shows that it is continuous elsewhere.

Example 2.3

Let $A = \mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$. Prove that the function

$$f(z) = \operatorname{Arg} z \quad (z \in A)$$

is continuous.

Solution

Let $\alpha \in A$ and suppose that $\varepsilon > 0$. We must find a number $\delta > 0$ such that

$$|f(z) - f(\alpha)| < \varepsilon, \quad \text{for all } z \in A \text{ with } |z - \alpha| < \delta.$$

We choose δ to satisfy two conditions. The first condition is that δ is less than the distance from α to the negative real axis; this ensures that the disc centred at α of radius δ lies entirely within A, as shown in the left-hand diagram of Figure 2.10 (so $|z - \alpha| < \delta \implies z \in A$).

The second condition is that δ must be small enough that the angle θ shown in the figure is less than ε. Applying trigonometry to the right-angled triangle shown in the figure, we see that $\sin\theta = \delta/|\alpha|$. So the condition $\theta < \varepsilon$ holds if we choose $\delta > 0$ such that $\sin^{-1}(\delta/|\alpha|) < \varepsilon$.

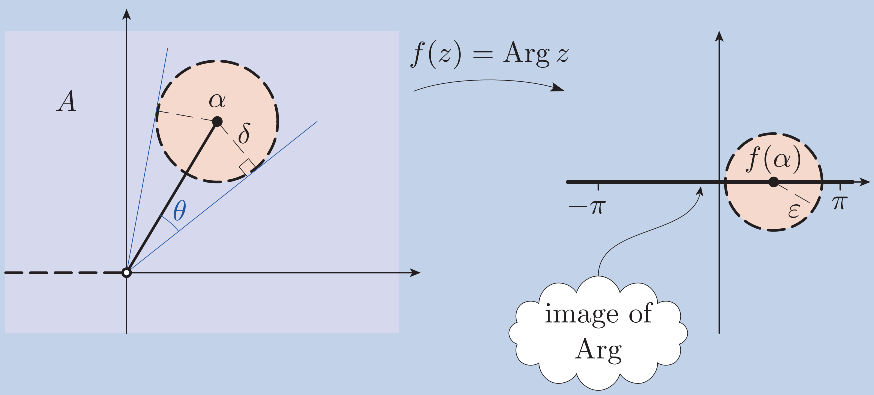


Figure 2.10 Continuity of $f(z) = \operatorname{Arg} z$ at α

If we choose δ to satisfy these two conditions, then we see that

$$|z - \alpha| < \delta \implies |\operatorname{Arg} z - \operatorname{Arg} \alpha| < \theta < \varepsilon.$$

Hence $f(z) = \operatorname{Arg} z$ is continuous on A.

Since $\operatorname{Arg} z$ appears in the definition of $\operatorname{Log} z$, and $\operatorname{Log} z$ is used to define z^α, where α is any complex number, it follows from the Composition Rule and the Combination Rules that $z \longmapsto \operatorname{Log} z$ and $z \longmapsto z^\alpha$ are also continuous on the set $\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$.

The following theorem collects together the continuous functions that we have met so far.

Theorem 2.4 Basic Continuous Functions

The following functions are continuous:

(a) polynomial and rational functions

(b) $f(z) = |z|$, $\overline{z}$, $\operatorname{Re} z$, $\operatorname{Im} z$

(c) $f(z) = e^z$

(d) trigonometric and hyperbolic functions

(e) $f(z) = \operatorname{Arg} z$, $\operatorname{Log} z$, z^α, on $\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$.

We have now seen several techniques for proving that a given function f is continuous at a point α in its domain:

1. Use the sequential definition.
2. Use the Combination Rules, Composition Rule or Restriction Rule, together with the list of basic continuous functions.
3. Use the ε-δ definition.

In most cases you will be able to verify that a given function f is continuous using the second technique – you should resort to the definitions only if you think that f is discontinuous at α, or if you can find no other way to prove that f is continuous at α (see Exercise 2.8).

Exercise 2.6

Prove that the following function is continuous:

$$f(z) = \sin\left(\frac{z^2+1}{z-2i}\right) \quad (z \in A),$$

where $A = \{z : -1 \leq \operatorname{Re} z \leq 1,\ -1 \leq \operatorname{Im} z \leq 1\}$.

We can use the fact that a function f is continuous at a point α to evaluate limits of sequences of the form $(f(z_n))$, where the sequence (z_n) has limit α. For it follows immediately from the definition of continuity that if the function f is continuous at α and $\lim_{n\to\infty} z_n = \alpha$, then

$$\lim_{n\to\infty} f(z_n) = f(\alpha).$$

Example 2.4

Evaluate $\lim_{n\to\infty} \operatorname{Log}(1 + i/n)$.

Solution

Since $(1/n)$ is a basic null sequence, we can apply the Combination Rules for sequences to see that the sequence

$$z_n = 1 + i/n, \quad n = 1, 2, \ldots,$$

has limit 1. Now, 1 is a point at which the function $f(z) = \operatorname{Log} z$ is continuous, by Theorem 2.4, so

$$\lim_{n\to\infty} \operatorname{Log}(1 + i/n) = \operatorname{Log} 1 = 0.$$

Exercise 2.7

Evaluate each of the following limits (of sequences).

(a) $\lim_{n\to\infty} \sin(\pi - 2/n)$ (b) $\lim_{n\to\infty} \operatorname{Arg}(i + 1/n^2)$

(c) $\lim_{n\to\infty} \exp(-i\pi/2 + i/(2n))$

Continuity: a 'local' property

Continuity is a property of a function at a point; that is, it is a 'local' property, in the following sense. Whether or not a function f is continuous at a point α depends only on the values taken by $f(z)$ when z lies in an open disc with centre α (no matter how small the radius of the disc) and in the domain of f. (This follows from either definition of continuity.)

As an application of this fact, consider the two functions

$$f(z) = z^{1/2} \quad (z \in \mathbb{C} - \{x \in \mathbb{R} : x \le 0\})$$

and

$$g(z) = z^{1/2} \quad (z \in \mathbb{C}).$$

Then g has the same rule as f, but a larger domain.

As stated in Theorem 2.4, the function f is continuous at each point of its domain $\mathbb{C} - \{x \in \mathbb{R} : x \le 0\}$, since

$$f(z) = \exp\left(\tfrac{1}{2} \operatorname{Log} z\right) \quad (z \in \mathbb{C} - \{x \in \mathbb{R} : x \le 0\}).$$

From this we deduce that the function g is also continuous at each point of $\mathbb{C} - \{x \in \mathbb{R} : x \le 0\}$. For if $\alpha \in \mathbb{C} - \{x \in \mathbb{R} : x \le 0\}$, then g takes the same value as f at each point of some open disc with centre α (one small enough that it does not meet the negative real axis), and the continuity of g at α then follows from that of f because continuity is a local property.

In the following exercise, you are asked to investigate the continuity of g at the other points of its domain $\mathbb{C}$.

Exercise 2.8

Show that the function

$$g(z) = z^{1/2} \quad (z \in \mathbb{C})$$

is

(a) discontinuous at each point $\alpha \in \{x \in \mathbb{R} : x < 0\}$

(b) continuous at 0, by using the ε-δ definition of continuity.

(*Hint*: For part (a), consider the sequence $z_n = |\alpha|e^{i(\pi+1/n)}$, $n = 1, 2, \ldots$, and use the result of Exercise 2.7(c).)

2.3 Proofs of continuity theorems

Here we prove the Combination Rules, the Composition Rule and the Restriction Rule, and we also prove that the sequential definition of continuity and the ε-δ definition of continuity are equivalent. This subsection may be omitted on a first reading.

Theorem 2.1 Combination Rules for Continuous Functions

If f and g are functions that are continuous at α, then

(a) **Sum Rule** $f + g$ is continuous at α

(b) **Multiple Rule** λf is continuous at α, for $\lambda \in \mathbb{C}$

(c) **Product Rule** fg is continuous at α

(d) **Quotient Rule** f/g is continuous at α, provided that $g(\alpha) \neq 0$.

The proofs of these four rules are similar to, and depend on, the corresponding results for sequences. We prove only the Sum Rule.

Proof

Sum Rule We want to prove that for each sequence (z_n) in the domain of $f + g$ such that $z_n \to \alpha$,

$$f(z_n) + g(z_n) \to f(\alpha) + g(\alpha).$$

We know that (z_n) lies in the domain of f and in the domain of g, and also that both f and g are continuous at α. Hence

$$f(z_n) \to f(\alpha) \quad \text{and} \quad g(z_n) \to g(\alpha),$$

and so, by the Sum Rule for sequences,

$$f(z_n) + g(z_n) \to f(\alpha) + g(\alpha), \quad \text{as required.} \qquad \blacksquare$$

Next recall the Composition Rule.

Theorem 2.2 Composition Rule

Let f be a function that is continuous at α, and let g be a function that is continuous at $f(\alpha)$. Then $g \circ f$ is continuous at α.

Proof We want to prove that for each sequence (z_n) in the domain of $g \circ f$ such that $z_n \to \alpha$,

$$g(f(z_n)) \to g(f(\alpha)).$$

We know that (z_n) lies in the domain of f, and that f is continuous at α. Hence

$$f(z_n) \to f(\alpha).$$

We also know that $(f(z_n))$ lies in the domain of g, and that g is continuous at $f(\alpha)$. Hence

$$g(f(z_n)) \to g(f(\alpha)), \text{ as required.}$$ ■

The proof of the next result is particularly straightforward.

Theorem 2.3 Restriction Rule for continuous functions

Let f and g be complex functions with domains A and B, respectively, and let $A \subseteq B$. If

- $f(z) = g(z)$, for $z \in A$
- g is continuous at $\alpha \in A$,

then f is continuous at α.

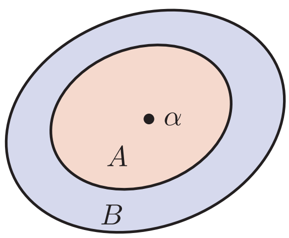


Figure 2.11 A point α in a set A contained in a set B

The sets A and B are illustrated in Figure 2.11.

Proof We want to prove that for each sequence (z_n) in A such that $z_n \to \alpha$,

$$f(z_n) \to f(\alpha).$$

Since (z_n) and α lie in B, $g(z_n) \to g(\alpha)$. Therefore, since

$$f(z_n) = g(z_n), \text{ for } n = 1, 2, \ldots, \quad \text{and} \quad f(\alpha) = g(\alpha),$$

we deduce that $f(z_n) \to f(\alpha)$, as required. ■

Finally, we prove the equivalence of our two definitions of continuity.

Theorem 2.5

The ε-δ definition of continuity is equivalent to the sequential definition of continuity.

Proof We give the proof in two parts: first we show that if f satisfies the ε-δ definition of continuity at $\alpha \in A$, then it also satisfies the sequential definition of continuity at α, and then we show that if f does *not* satisfy the ε-δ definition of continuity at α, then it does not satisfy the sequential definition of continuity at α.

1. Assume first that f satisfies the ε-δ definition of continuity at α. We want to deduce that if (z_n) is *any* sequence in A such that $z_n \to \alpha$, then $f(z_n) \to f(\alpha)$ (so f satisfies the sequential definition of continuity at α).

   Suppose, therefore, that $\varepsilon > 0$ is given. By our assumption, there is a $\delta > 0$ such that
   $$|f(z) - f(\alpha)| < \varepsilon, \quad \text{for all } z \in A \text{ with } |z - \alpha| < \delta.$$
   Furthermore, there is an integer N such that
   $$|z_n - \alpha| < \delta, \quad \text{for all } n > N.$$
   Hence
   $$|f(z_n) - f(\alpha)| < \varepsilon, \quad \text{for all } n > N.$$
   Thus $f(z_n) \to f(\alpha)$, as required.

2. Next assume that f does *not* satisfy the ε-δ definition of continuity at α. We want to find a sequence (z_n) in A with $z_n \to \alpha$ but $f(z_n) \not\to f(\alpha)$ (so f does not satisfy the sequential definition of continuity at α).

   By our assumption, there is some number $\varepsilon > 0$ such that, for *each* $\delta > 0$,
   $$|f(z) - f(\alpha)| \geq \varepsilon, \quad \text{for some } z \in A \text{ with } |z - \alpha| < \delta.$$
   Applying this statement with $\delta = 1/n, n = 1, 2, \ldots$, we find that
   $$|f(z_n) - f(\alpha)| \geq \varepsilon, \quad \text{for some } z_n \in A \text{ with } |z_n - \alpha| < 1/n.$$
   The numbers z_n, $n = 1, 2, \ldots$, form a sequence in A with $z_n \to \alpha$. However, $f(z_n) \not\to f(\alpha)$ because all the terms of the sequence $(f(z_n))$ lie outside the circle with centre $f(\alpha)$ and radius ε. This completes the proof of the theorem. ■

Further exercises

Exercise 2.9

For each of the following functions, decide whether it is continuous or discontinuous at the point α and then use the appropriate strategy, based on the definition of continuity, to prove your decision.

(a) $f(z) = z^2, \quad \alpha = 2i$ (b) $f(z) = z^{1/3}, \quad \alpha = -1$

Exercise 2.10

Using the Combination Rules, the Composition Rule, the Restriction Rule and the list of basic continuous functions in Theorem 2.4, prove that each of the following functions is continuous.

(a) $f(z) = 3z^3 + |z| \operatorname{Re} z$

(b) $f(z) = |\sin z|$

(c) $f(x) = 1 + x(i-1) \quad (x \in [0,1])$

(d) $f(x) = \cos x + i \sin x \quad (x \in [0, 2\pi])$

Exercise 2.11

Consider the function $f(z) = \theta$, where θ is the argument of z that lies in the interval $[0, 2\pi)$. Write down the domain of f, and prove that f is discontinuous at 1.

Exercise 2.12

Evaluate each of the following limits.

(a) $\lim_{n\to\infty} \operatorname{Log}(\pi + i/n)$

(b) $\lim_{n\to\infty} \exp\left(\dfrac{(2n+1)\pi}{2n-1} i\right)$

(c) $\lim_{n\to\infty} \cos\left(\dfrac{(1+i)^n}{(2+i)^n}\right)$

Bernard Bolzano

Origins of continuity

One of the first to formulate the concept of continuity in a manner that is close to modern definitions was the mathematician and philosopher Bernard Bolzano (1781–1848), who was born in Prague. Bolzano wrote, in 1817:

> A function $f(x)$ varies according to the law of continuity for all values of x inside or outside certain limits ... if x is some such value, the difference $f(x + \omega) - f(x)$ can be made smaller than any given quantity provided ω can be taken as small as we please.
>
> (Katz, 1993, p. 641)

Apparently independently, the prolific French mathematician Augustin-Louis Cauchy (1789–1857) published a similar definition four years later in his seminal textbook on analysis, *Cours d'Analyse* (1821), cited in Katz (1993, p. 641):

> The function $f(x)$ will be, between two assigned values of the variable x, a continuous function of this variable if for each value of x between these limits, the [absolute] value of the difference $f(x+\alpha)-f(x)$ decreases indefinitely with α.

Both definitions are similar to the ε-δ definition of continuity introduced in this section, although Cauchy attempts to define continuity on a whole interval at once, rather than at a single point, as we do (and as all modern texts do). The two mathematicians were, at first, concerned with real functions, but Cauchy also considered continuity of functions of several variables and complex functions. In fact, Cauchy went on to make tremendous contributions to the theory of complex functions, and you will encounter his name many times in this module.

Augustin-Louis Cauchy

3 Limits of functions

After working through this section, you should be able to:

- show that a point α is a *limit point* of a set A
- explain the statement 'the function f has limit β as z tends to α', and evaluate such a limit by working from the definition
- establish that certain functions do not have limits at specified points
- understand the relationship between limits and continuity, and use it to evaluate certain limits of functions
- understand the Combination Rules for limits of functions.

3.1 Defining the limit of a function

We begin by looking in detail at the rational function

$$f(z)=\frac{z^2+1}{z-i}.$$

The domain of this function is $\mathbb{C}-\{i\}$ because the denominator $z-i$ equals zero when $z=i$. However, the function f is actually rather well-behaved as z approaches this missing point i. To see why this is the case, notice that

$$f(z)=\frac{z^2+1}{z-i}=\frac{(z-i)(z+i)}{z-i}=z+i,\quad \text{for } z\in\mathbb{C}-\{i\}.$$

The factor $z-i$ can be cancelled here because $z\in\mathbb{C}-\{i\}$, so $z-i\neq 0$. This equivalent formula for $f(z)$ makes it clear that if z tends to i, then $f(z)$ tends to $i+i=2i$.

The behaviour of this function f near i is an example of a function *tending to a limit* as z tends to a point α. In order to define this concept for a

general function f, we need to ensure that f is defined *near* the point α, but not necessarily *at* the point α. To do this, we introduce the idea of a *limit point* of a set A.

Definition

The point α is a **limit point** of a set A in $\mathbb{C}$ if there is a sequence (z_n) in $A - \{\alpha\}$ such that

$$\lim_{n\to\infty} z_n = \alpha.$$

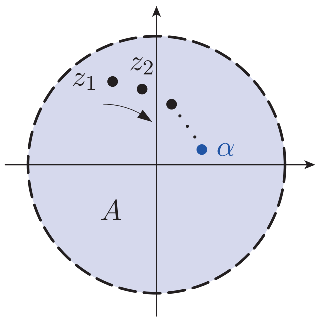


Figure 3.1 A limit point inside $A = \{z : |z| < 1\}$

Note that some texts use the phrase *cluster point* or *accumulation point* in place of limit point.

If α is a limit point of A, then it is possible to approach α along a sequence (z_n) of points, each of which belongs to A but none of which is equal to α. As an example, consider the open disc

$$A = \{z : |z| < 1\}.$$

Figure 3.1 indicates that each point α of A is a limit point of A, but it is also true that each boundary point of A is a limit point of A. For example, consider $\alpha = i$; all points of the sequence

$$z_n = i - i/n, \quad n = 1, 2, \ldots,$$

lie in A and are different from i (Figure 3.2), but $\lim_{n\to\infty} z_n = i$, so i is a limit point of A.

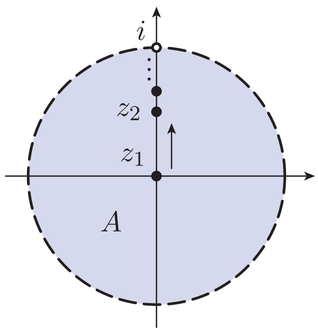


Figure 3.2 A limit point on the boundary of $A = \{z : |z| < 1\}$

On the other hand, if $A = \mathbb{Z}$, then $\alpha = 1$ belongs to A, but α is not a limit point of A; in fact, the set $A = \mathbb{Z}$ has no limit points at all.

Exercise 3.1

Prove that each of the following points α is a limit point of the corresponding set A.

(a) $\alpha = 0$, $A = \{z : |z| < 1\}$

(b) $\alpha = i$, $A = \{z : \operatorname{Re} z > 0\}$

(c) $\alpha = 1$, $A = \{z : |z| = 1\}$

(d) $\alpha = 2$, $A = \mathbb{C} - \{2\}$

(e) $\alpha = -1$, $A = \mathbb{R} - \{-1\}$

We now define the *limit of a function* using the convergence of sequences, in a manner reminiscent of the definition of continuity.

Definitions

Let f be a function with domain A, and suppose that α is a limit point of A. Then the function f has **limit β as z tends to α** if, for each sequence (z_n) in $A - \{\alpha\}$ such that $z_n \to \alpha$, we have

$$f(z_n) \to \beta.$$

In this case we write

- *either* $\lim\limits_{z\to\alpha} f(z) = \beta$
- *or* $f(z) \to \beta$ as $z \to \alpha$,

and we say that the **limit exists**.

Observe that the *limit point* α is associated with the set A, whereas the *limit* β is associated with the function f.

Figure 3.3 illustrates this definition.

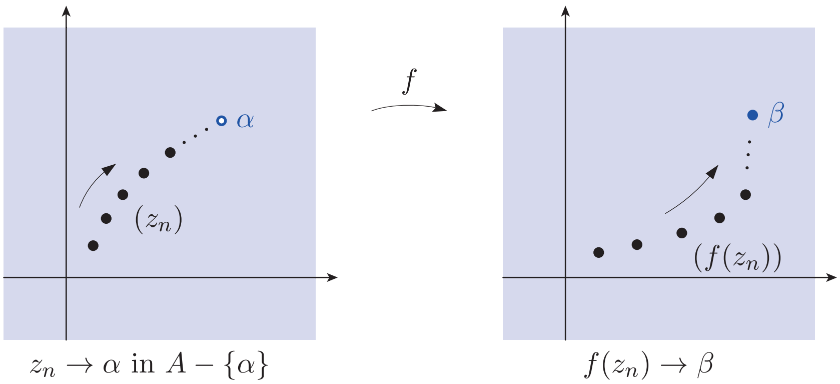


Figure 3.3 Convergent sequences (z_n) and $(f(z_n))$

Remarks

1. Since the sequences (z_n) considered here lie in $A - \{\alpha\}$, the value $f(\alpha)$ need not be defined in order for $\lim\limits_{z\to\alpha} f(z)$ to exist. Even when $f(\alpha)$ is defined, its value has no bearing on the existence or the value of this limit.
2. The expression $\lim\limits_{z\to\alpha} f(z) = \beta$ contains no reference to any sequence; this signifies that the limit is independent of the route taken to α through $A - \{\alpha\}$.

Earlier you saw that the function $f(z) = (z^2+1)/(z-i)$ can be written in the form

$$f(z) = \frac{z^2+1}{z-i} = z + i \quad (z \in \mathbb{C} - \{i\}).$$

Thus if (z_n) is any sequence in $\mathbb{C} - \{i\}$ such that $z_n \to i$, then

$$f(z_n) = z_n + i \to i + i = 2i,$$

by the Sum Rule for sequences. Hence $\lim\limits_{z\to i} f(z) = 2i$, as expected.

Example 3.1

Prove that the following limit exists, and evaluate it:

$$\lim_{z \to 2} \frac{z^2 - 4}{z - 2}.$$

Solution

First, note that the domain of the function

$$f(z) = \frac{z^2 - 4}{z - 2}$$

is $\mathbb{C} - \{2\}$ and that 2 is a limit point of this set (by Exercise 3.1(d)).

Next, notice that

$$f(z) = \frac{z^2 - 4}{z - 2} = z + 2, \quad \text{for } z \in \mathbb{C} - \{2\}.$$

Thus if (z_n) is any sequence lying in $\mathbb{C} - \{2\}$ such that $z_n \to 2$, then

$$f(z_n) = z_n + 2 \to 4,$$

by the Sum Rule for sequences. Hence

$$\lim_{z \to 2} \frac{z^2 - 4}{z - 2} = 4.$$

Exercise 3.2

Prove that the following limit exists, and evaluate it:

$$\lim_{z \to i} \frac{z^3 + i}{z - i}.$$

In the next two subsections, you will learn alternative methods for evaluating limits such as those in Example 3.1 and Exercise 3.2.

Our next example illustrates how to prove that a limit does not exist, by showing that there is no β such that $f(z_n) \to \beta$ for *every* sequence (z_n) that tends to α through $A - \{\alpha\}$.

Example 3.2

Prove that neither of the following limits exists.

(a) $\lim_{z \to 0} z/\overline{z}$ (b) $\lim_{z \to 0} 1/z$

Solution

(a) The function $f(z) = z/\overline{z}$ has domain $\mathbb{C} - \{0\}$ and 0 is a limit point of this set. However,

$$z = x,\ x \in \mathbb{R} - \{0\} \implies f(z) = f(x) = x/x = 1,$$
$$z = iy,\ y \in \mathbb{R} - \{0\} \implies f(z) = f(iy) = (iy)/(-iy) = -1.$$

This shows that $f(z)$ behaves in different ways as z approaches 0 in different directions. To exploit this, let us choose simple sequences tending to 0 in the two directions, namely

$$z_n = \frac{1}{n} \quad \text{and} \quad z_n' = \frac{i}{n}, \quad n = 1, 2, \ldots.$$

Then both sequences (z_n) and (z_n') lie in the domain $\mathbb{C} - \{0\}$ and both tend to 0, but

$$\lim_{n\to\infty} f(z_n) = 1, \quad \text{whereas} \quad \lim_{n\to\infty} f(z_n') = -1.$$

This is summarised in Figure 3.4.

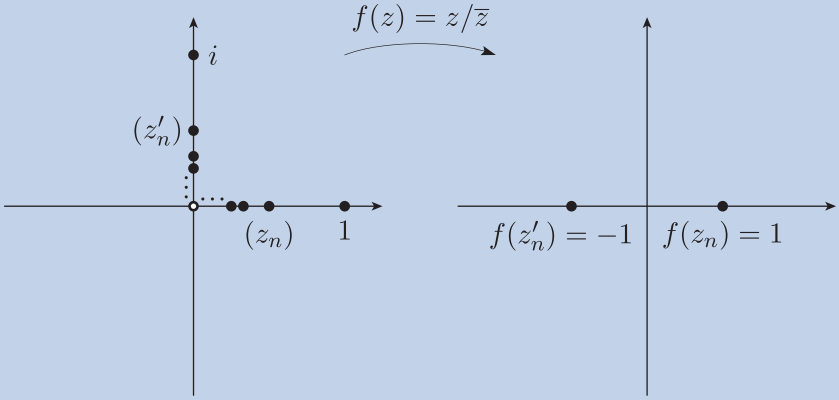


Figure 3.4 The function $f(z) = z/\overline{z}$ maps every term of (z_n) to 1 and it maps every term of (z_n') to -1

Hence there is no number β such that $f(z_n) \to \beta$ for each sequence (z_n) that tends to 0 through $\mathbb{C} - \{0\}$, so f does not tend to a limit as $z \to 0$.

(b) The domain of $f(z) = 1/z$ is $\mathbb{C} - \{0\}$ and 0 is a limit point of this set. However, the sequence $z_n = 1/n$, $n = 1, 2, \ldots$, lies in $\mathbb{C} - \{0\}$ and

$$f(z_n) = 1/z_n = n \to \infty \text{ as } n \to \infty.$$

Hence there is no number β such that $f(z_n) \to \beta$ for each sequence (z_n) that tends to 0 through $\mathbb{C} - \{0\}$, so f does not tend to a limit as z tends to 0.

Remark

In fact, $1/z \to \infty$ as $z \to 0$. We discuss this type of behaviour later in the module.

We now summarise these techniques in the form of alternative strategies.

Strategy for proving that a limit does not exist

To prove that $\lim\limits_{z\to\alpha} f(z)$ does not exist, where α is a limit point of the domain A of the function f:

- *either* find two sequences (z_n) and (z'_n) in $A - \{\alpha\}$ that both tend to α such that the sequences $(f(z_n))$ and $(f(z'_n))$ have different limits
- *or* find a sequence (z_n) in $A - \{\alpha\}$ that tends to α such that the sequence $(f(z_n))$ tends to infinity.

Notice the similarity of this strategy to Theorem 1.6, the Subsequence Rules.

Exercise 3.3

Prove that $\lim\limits_{z\to 0}(z/\operatorname{Re} z)$ does not exist.

3.2 Limits and continuity

Comparison of the definition of *limit* given in this section to the definition of *continuity* given in the previous section shows that there is a close connection between these notions. The following result makes this connection precise.

Theorem 3.1

Let f be a function with domain A and suppose that the point $\alpha \in A$ is a limit point of A. Then

$$f \text{ is continuous at } \alpha \iff \lim_{z\to\alpha} f(z) = f(\alpha).$$

Proof Suppose first that f is continuous at α. If (z_n) is any sequence in $A - \{\alpha\}$ with $z_n \to \alpha$, then it follows (because f is continuous at α) that $f(z_n) \to f(\alpha)$. Hence $\lim\limits_{z\to\alpha} f(z) = f(\alpha)$, as required.

Suppose next that $\lim\limits_{z\to\alpha} f(z) = f(\alpha)$. Let (z_n) be any sequence in A with $z_n \to \alpha$. We can separate (z_n) into two subsequences:

(z_{m_k}), the subsequence of terms with $f(z_{m_k}) = f(\alpha)$, $k = 1, 2, \dots$

(z_{n_k}), the subsequence of terms with $f(z_{n_k}) \neq f(\alpha)$, $k = 1, 2, \dots$.

Clearly, $f(z_{m_k}) \to f(\alpha)$. And $f(z_{n_k}) \to f(\alpha)$ also, because $z_{n_k} \to \alpha$ and $\lim_{z\to\alpha} f(z) = f(\alpha)$. Thus $(f(z_n))$ can be separated into two subsequences, each of which converges to $f(\alpha)$. Therefore, as we saw in Exercise 1.14,

$$f(z_n) \to f(\alpha);$$

hence f is continuous at α. ■

Theorem 3.1 gives us a method for evaluating some of the limits from the previous subsection. For example, let us return to the limit

$$\lim_{z\to i} \frac{z^2+1}{z-i}$$

considered earlier. You saw that

$$\frac{z^2+1}{z-i} = z+i, \quad \text{for } z \in \mathbb{C} - \{i\}.$$

The polynomial function $f(z) = z + i$ is continuous on $\mathbb{C}$, and i is a limit point of $\mathbb{C}$, so Theorem 3.1 tells us that

$$\lim_{z\to i} \frac{z^2+1}{z-i} = \lim_{z\to i} f(z) = f(i) = 2i.$$

The next exercise asks you to revisit the limits from Example 3.1 and Exercise 3.2 using Theorem 3.1.

Exercise 3.4

Using Theorem 3.1, prove that the following limits exist, and evaluate them.

(a) $\displaystyle\lim_{z\to 2} \frac{z^2-4}{z-2}$ (b) $\displaystyle\lim_{z\to i} \frac{z^3+i}{z-i}$

Just as for continuity, the definition of limit can be reformulated in ε-δ terms, as follows.

Definition

Let $f\colon A \longrightarrow \mathbb{C}$ and suppose that α is a limit point of A. Then the function f has **limit $\boldsymbol{\beta}$ as $\boldsymbol{z}$ tends to $\boldsymbol{\alpha}$** if, for each $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|f(z) - \beta| < \varepsilon, \quad \text{for all } z \in A - \{\alpha\} \text{ with } |z - \alpha| < \delta.$$

The proof that this definition is equivalent to the earlier sequential definition is similar to that of the corresponding result for continuity (Theorem 2.5).

3.3 Rules for limits

As you might expect (from your experience with limits of sequences and continuous functions), limits can often be evaluated by applying rules for combining limits.

Theorem 3.2 Combination Rules for Limits of Functions

Let f and g be functions with domains A and B, respectively, and suppose that α is a limit point of $A \cap B$. If

$$\lim_{z\to\alpha} f(z) = \beta \quad \text{and} \quad \lim_{z\to\alpha} g(z) = \gamma,$$

then

(a) **Sum Rule** $\lim_{z\to\alpha}(f(z) + g(z)) = \beta + \gamma$

(b) **Multiple Rule** $\lim_{z\to\alpha}(\lambda f(z)) = \lambda\beta$, for $\lambda \in \mathbb{C}$

(c) **Product Rule** $\lim_{z\to\alpha}(f(z)g(z)) = \beta\gamma$

(d) **Quotient Rule** $\lim_{z\to\alpha}(f(z)/g(z)) = \beta/\gamma$, provided that $\gamma \neq 0$.

Implicit in the statement of these rules is the observation that if α is a limit point of the intersection of sets $A \cap B$, then clearly it is also a limit point of A and of B.

The proofs of the Combination Rules are straightforward and depend on the analogous results for sequences (see Theorem 1.3); we omit them.

To calculate limits, these Combination Rules can often be used in place of the approach from the previous subsection based on Theorem 3.1. Throughout this module we use whichever approach seems most convenient.

Further exercises

Exercise 3.5

For each of the following points α and sets A, decide whether α is a limit point of A. In those cases where α is a limit point of A, prove it by specifying a suitable sequence (z_n) in $A - \{\alpha\}$ that converges to α.

(a) $\alpha = i$, $A = \{z : \operatorname{Re} z > 1\}$

(b) $\alpha = 1$, $A = \{z : \operatorname{Re} z + \operatorname{Im} z = 1\}$

Exercise 3.6

Determine whether or not each of the following limits (of functions) exists. For each one that does, evaluate the limit.

(a) $\displaystyle\lim_{z\to 3}\frac{z^3-27}{z-3}$ (b) $\displaystyle\lim_{z\to -i}\frac{z^2+1}{z+i}$ (c) $\displaystyle\lim_{z\to i\pi}\left(e^z\sinh z+\frac{1}{z}\right)$

(d) $\displaystyle\lim_{z\to 1}\frac{1}{\operatorname{Im} z}$ (e) $\displaystyle\lim_{z\to i}\frac{\operatorname{Re} z}{\operatorname{Im} z}$ (f) $\displaystyle\lim_{z\to 0}\frac{\operatorname{Re} z}{\operatorname{Im} z}$

4 Regions

After working through this section, you should be able to:

- determine whether a given subset of $\mathbb{C}$ is *open*, *convex*, *connected*
- explain the statement '$\mathcal{R}$ is a region of $\mathbb{C}$'
- recognise certain *basic regions*.

4.1 Domains of complex functions

Various sets occur as the domains of complex functions. For example,

the function $f(z)=\sin z$ has domain $\mathbb{C}$
the function $f(z)=1/z$ has domain $\mathbb{C}-\{0\}$
the function $f(z)=\tan z$ has domain $\mathbb{C}-\left\{\left(n+\frac{1}{2}\right)\pi : n\in\mathbb{Z}\right\}$

(see Figure 4.1).

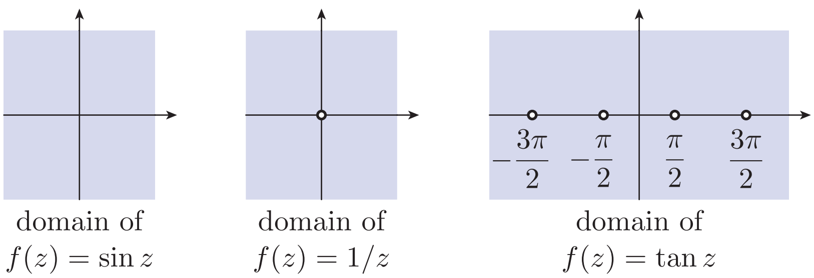


Figure 4.1 Domains of various functions

Although these sets are all different, they have certain features in common. For example, each of the sets is 'unbounded'; that is, they do not lie inside any circle in $\mathbb{C}$.

A less obvious property of these sets is the following: each point in the set can be surrounded by an open disc lying entirely within the set. For example, the point 1 in $\mathbb{C}-\{0\}$ is the centre of the open disc $\left\{z:|z-1|<\frac{1}{2}\right\}$ which lies entirely in $\mathbb{C}-\{0\}$ (Figure 4.2). Sets with this property are called *open*.

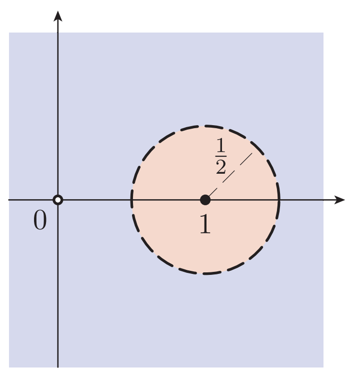


Figure 4.2 Open disc centred at 1 lying in $\mathbb{C}-\{0\}$

Another property of these sets is the following: each pair of points in the set can be joined by a path lying entirely in the set. For example, as shown

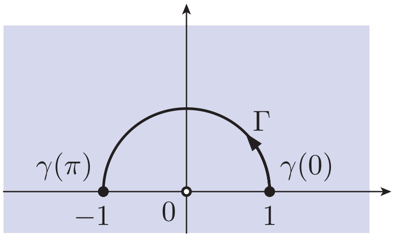


Figure 4.3 Points 1 and −1 connected by a path in $\mathbb{C} - \{0\}$

in Figure 4.3, the points 1 and -1 in $\mathbb{C} - \{0\}$ can be joined by the semicircular path Γ with parametrisation $\gamma(t) = e^{it}$ $(t \in [0, \pi])$. Sets with this property are called *connected*.

It turns out that many functions have domains that are both open and connected and that these two properties are useful. We therefore devote the rest of this section to a discussion of these properties.

4.2 Open sets

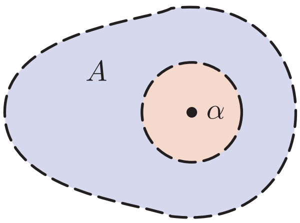


Figure 4.4 An open set A

In Unit A1 we defined open discs and open half-planes. In both cases, no boundary points are included. More generally, an *open set* can be thought of as a set that does not include any of its boundary points. The following definition, however (illustrated in Figure 4.4), does not mention the boundary.

> **Definition**
>
> A set A in $\mathbb{C}$ is **open** if each point α in A is the centre of some open disc lying entirely in A.

For example, the disc $D = \{z : |z| < 1\}$ is open because each point $\alpha \in D$ lies at a distance $1 - |\alpha| > 0$ from the boundary of D, so the open disc

$$\{z : |z - \alpha| < 1 - |\alpha|\}$$

lies entirely in D (Figure 4.5(a)).

However, the disc $E = \{z : |z| \leq 1\}$ is *not* open; for example, the definition fails at $\alpha = 1$, since no open disc of the form

$$\{z : |z - 1| < r\}, \quad r > 0,$$

lies entirely in E (Figure 4.5(b)).

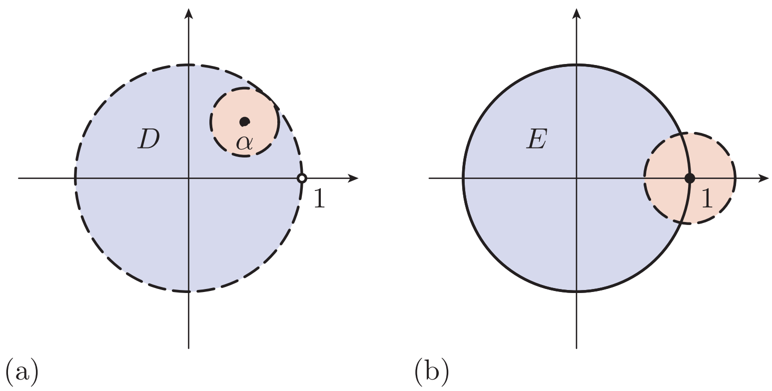


Figure 4.5 (a) D is open (b) E is not open

Similar arguments show that every open disc is an open set and every closed disc is not an open set.

Also, note that every singleton set $\{\alpha\}$ is not open: no open disc centred at α lies entirely in $\{\alpha\}$.

Example 4.1

Prove that each of the following sets is open.

(a) $\{z : \operatorname{Re} z > 0\}$

(b) $\{z : |z| > 1\}$

(c) $\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$

Solution

(a) Let $A = \{z : \operatorname{Re} z > 0\}$, so the boundary of A is $\{z : \operatorname{Re} z = 0\}$. If $\alpha \in A$, then $\alpha = a + ib$, where $a > 0$, and the distance from α to the boundary of A is a. Hence the open disc

$$\{z : |z - \alpha| < a\}$$

lies entirely in A (Figure 4.6), so A is open.

It follows similarly that every open half-plane is an open set.

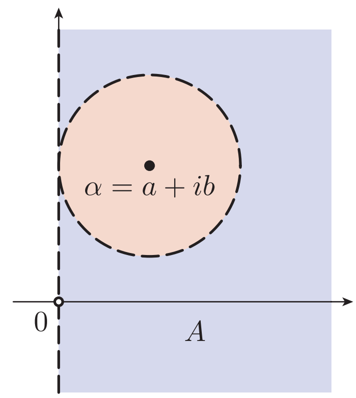


Figure 4.6 Open disc centred at α lying in $A = \{z : \operatorname{Re} z > 0\}$

(b) Let $A = \{z : |z| > 1\}$, so the boundary of A is $\{z : |z| = 1\}$. If $\alpha \in A$, then $|\alpha| > 1$ and the distance from α to the boundary of A is $|\alpha| - 1 > 0$. Hence the open disc

$$\{z : |z - \alpha| < |\alpha| - 1\}$$

lies entirely in A (Figure 4.7), so A is open.

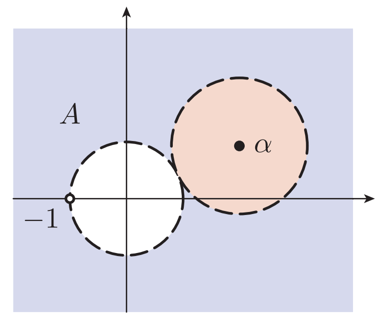


Figure 4.7 Open disc centred at α lying in $A = \{z : |z| > 1\}$

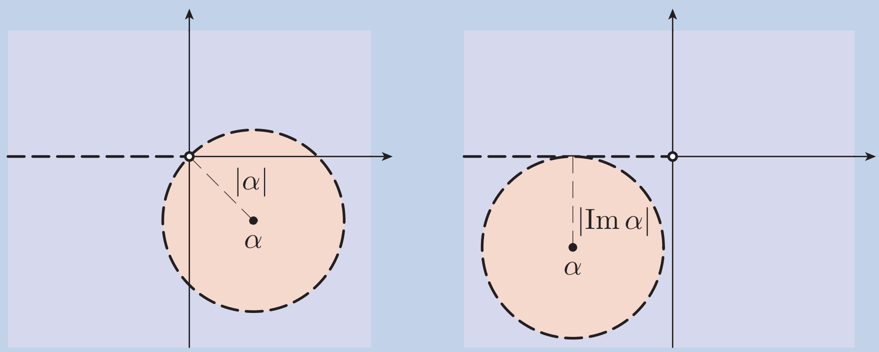
(c) Let $A = \mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$, so the boundary of A is $\{x \in \mathbb{R} : x \leq 0\}$. If $\alpha \in A$, then the distance from α to the boundary of A is

$$r_\alpha = \begin{cases} |\alpha|, & \operatorname{Re} \alpha \geq 0, \\ |\operatorname{Im} \alpha|, & \operatorname{Re} \alpha < 0. \end{cases}$$

Hence the open disc

$$\{z : |z - \alpha| < r_\alpha\}$$

lies entirely in A (Figure 4.8), so A is open.



(a) (b)

Figure 4.8 Open discs centred at α lying in $A = \mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$ in the two cases (a) $\operatorname{Re} \alpha \geq 0$, (b) $\operatorname{Re} \alpha < 0$

In each of these solutions, we chose the radius of the disc with centre α to be as large as possible, although there is no particular reason for doing this. For example, in Example 4.1(a), we could have taken the radius to be $a/2$.

Exercise 4.1

Prove that each of the following sets is open.

(a) $\mathbb{C} - \{0\}$

(b) $\{z : -2 < \operatorname{Re} z < 2,\ -1 < \operatorname{Im} z < 1\}$

(c) $\{z : 1 < |z| < 2\}$

(d) $\{z : \pi/3 < \operatorname{Arg} z < 2\pi/3\}$

You have now seen a number of different types of basic open sets: open discs, open half-planes, complements of closed discs, open annuli, open rectangles and open sectors. In addition, $\mathbb{C}$ itself is open. We note that the empty set $\varnothing$ is also considered to be open, because it contains no point α where the definition fails.

Now we describe two ways of creating 'new open sets from old'.

Theorem 4.1 Combination Rules for Open Sets

If A_1 and A_2 are open sets, then so are

(a) $A_1 \cup A_2$

(b) $A_1 \cap A_2$.

Proof

(a) If $\alpha \in A_1 \cup A_2$, then α lies in A_1 or A_2 (or both). Let us assume that $\alpha \in A_1$; the case $\alpha \in A_2$ can be dealt with in a similar manner.

Since A_1 is open, there is a positive radius r such that

$$\{z : |z - \alpha| < r\} \subseteq A_1,$$

as indicated in Figure 4.9. Hence

$$\{z : |z - \alpha| < r\} \subseteq A_1 \cup A_2,$$

as required.

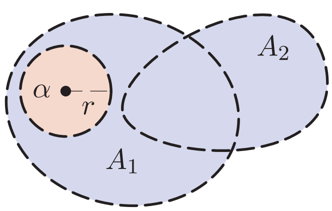


Figure 4.9 Open disc centred at α lying in A_1

(b) If $\alpha \in A_1 \cap A_2$, then α lies in both A_1 and A_2. Since A_1, A_2 are both open, there are positive radii r_1, r_2 such that

$$\{z : |z - \alpha| < r_1\} \subseteq A_1 \quad \text{and} \quad \{z : |z - \alpha| < r_2\} \subseteq A_2,$$

as indicated in Figure 4.10. Thus if $r = \min\{r_1, r_2\}$, then

$$\{z : |z - \alpha| < r\} \subseteq A_1 \cap A_2,$$

as required. ■

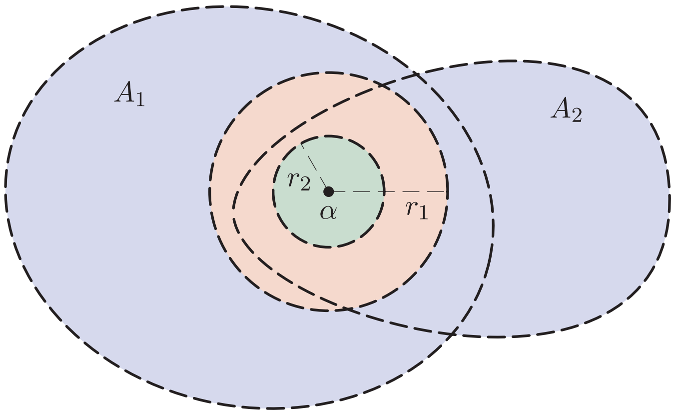


Figure 4.10 Open disc centred at α lying in A_1 and A_2

Applying the Principle of Mathematical Induction to Theorem 4.1, we obtain the following corollary.

Corollary

If $A_1, A_2, \ldots, A_n$ are open sets, then so are

(a) $A_1 \cup A_2 \cup \cdots \cup A_n$

(b) $A_1 \cap A_2 \cap \cdots \cap A_n$.

Remark

The proof of Theorem 4.1(a) can be readily adapted to show that the union of any (possibly infinite) collection of open sets is open. However, the intersection of an infinite collection of open sets need not be open. For example, if

$$A_n = \{z : |z| < 1/n\}, \quad n = 1, 2, \ldots,$$

then each A_n is open, but

$$\begin{aligned} A_1 \cap A_2 \cap \cdots &= \{z : z \in A_n, \text{ for } n = 1, 2, \ldots\} \\ &= \{0\}, \end{aligned}$$

which is not open.

Exercise 4.2

Use Theorem 4.1 or its corollary to prove that each of the following sets is open.

(a) $\{z : \operatorname{Re} z > 0,\ \operatorname{Im} z > 0,\ |z| < 1\}$

(b) $\{z : \operatorname{Re} z \neq 0\}$

4.3 Connected sets

In Unit A2 we defined a path to be a subset Γ of $\mathbb{C}$ that is the image of an associated continuous function $\gamma\colon I \longrightarrow \mathbb{C}$ (the parametrisation of Γ), where I is an interval of the real line; so $\Gamma = \gamma(I)$. If $I = [a, b]$, then $\gamma(a)$ and $\gamma(b)$ are, respectively, called the initial point and final point of the path. The path Γ is said to **join** $\gamma(a)$ to $\gamma(b)$.

Now we use paths to define the notion of a *connected set*.

Definition

A set A in $\mathbb{C}$ is (**pathwise**) **connected** if any two distinct points α and β in A can be joined by a path lying entirely in A.

The word 'pathwise' is usually omitted from the definition of a connected set. It is included in parentheses here because there is a more general notion of connectedness (which we do not require at this stage).

Figure 4.11 illustrates the definition of connectedness. Note that the path shown in the figure avoids the 'hole' and so lies entirely in A.

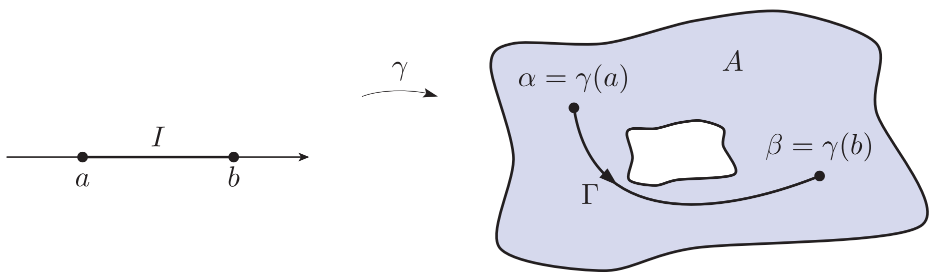


Figure 4.11 Path from α to β in the connected set A

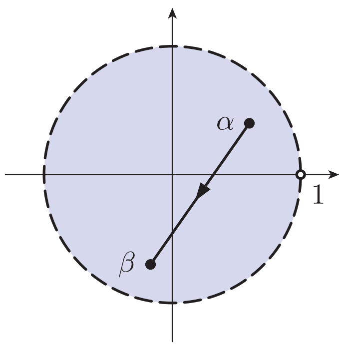


Figure 4.12 Line segment from α to β in a disc

The open disc $D = \{z : |z| < 1\}$ is connected because any two points $\alpha, \beta \in D$ can be joined by, for example, the line segment from α to β, which lies entirely in D (Figure 4.12). Similarly, the closed disc $E = \{z : |z| \leq 1\}$ is connected.

A connected set like one of the discs D and E in which any two distinct points α and β can be joined by a line segment that lies entirely within the set is called **convex**. Other examples of sets that are convex, and therefore connected, are any disc, half-plane or rectangle, and $\mathbb{C}$ itself.

However, not all connected sets are convex, as the following example shows.

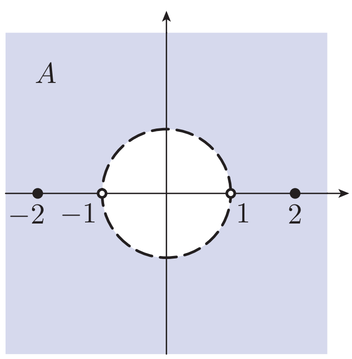


Figure 4.13 The set $A = \{z : |z| > 1\}$

Example 4.2

Prove that the set $\{z : |z| > 1\}$ is not convex, but is connected.

Solution

The set $A = \{z : |z| > 1\}$ is not convex because, for example, the interval $[-2, 2]$ is not a subset of A (Figure 4.13). However, any two

points α, β of A can be joined by a path in A. For example, if $|\alpha| = |\beta|$, then such a path Γ could be a circular arc, with centre 0, from α to β. If $|\alpha| \neq |\beta|$, then a suitable path Γ would consist of the (anticlockwise) circular arc, with centre 0, from α to $(|\alpha|/|\beta|)\beta$, followed by part of the ray from 0 through β (Figure 4.14).

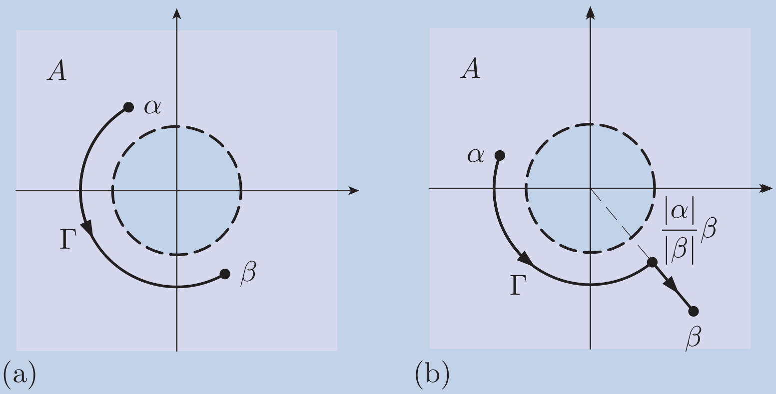


Figure 4.14 Paths in A joining α to β in the two cases (a) $|\alpha| = |\beta|$, (b) $|\alpha| \neq |\beta|$

Remarks

1. In this solution we did not attempt to construct the simplest or shortest path from α to β. Rather, we chose a reasonably simple *type of path* which works for any pair α, β in A. Many other choices of path could be made.
2. Although we did not specify the parametrisation of the path Γ joining α to β, this could be done if necessary. Suppose, for example, that $\alpha = 2i$ and $\beta = -4i$. Then the path

$$\Gamma_1 : \gamma_1(t) = 2e^{i\pi(t+1/2)} \quad (t \in [0,1])$$

has initial point $\gamma_1(0) = 2i$ and final point $\gamma_1(1) = -2i$, and the path

$$\Gamma_2 : \gamma_2(t) = -2ti \quad (t \in [1,2])$$

has initial point $\gamma_2(1) = -2i$ and final point $\gamma_2(2) = -4i$. Thus the path $\Gamma = \Gamma_1 \cup \Gamma_2$ with parametrisation

$$\gamma(t) = \begin{cases} \gamma_1(t), & t \in [0,1], \\ \gamma_2(t), & t \in [1,2], \end{cases}$$

joins $2i$ to $-4i$ (Figure 4.15). The continuity of γ follows from that of γ_1 and γ_2, together with the fact that $\gamma_1(1) = \gamma_2(1)$.

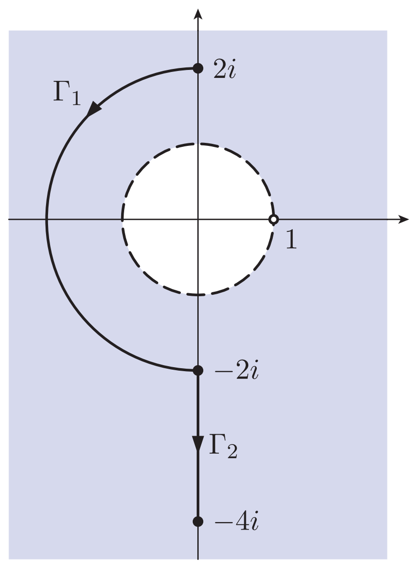


Figure 4.15 A path in $\{z : |z| > 1\}$ joining $2i$ to $-4i$

Notice that we chose the domains of γ_1 and γ_2 to be adjacent intervals so that the domain of γ is an interval (as required by the definition of a path). It may require some ingenuity to 'glue' two or more parametrisations together in this way. Usually, a geometric description of such a path is sufficient.

3. The solution to Example 4.2 can be readily adapted to prove the connectedness of other sets with rotational symmetry, such as $\mathbb{C} - \{0\}$ or an annulus.

Exercise 4.3

Prove that each of the following sets is connected.

(a) $\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$ (b) $\{z : -1 < \operatorname{Re} z < 1 \text{ or } -1 < \operatorname{Im} z < 1\}$

Is either of these sets convex?

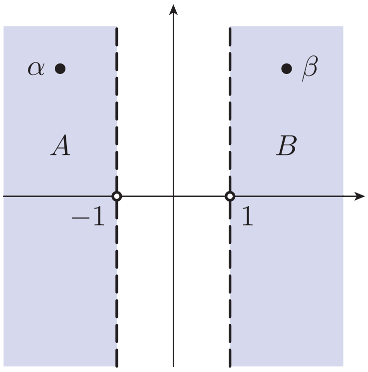


Figure 4.16 $A \cup B$ not connected

The property of connectedness is not preserved by forming unions or intersections. For example, if

$$A = \{z : \operatorname{Re} z < -1\} \quad \text{and} \quad B = \{z : \operatorname{Re} z > 1\},$$

then A and B are connected, but $A \cup B$ is not connected, since no point α in A can be joined to a point β in B by a path lying entirely in $A \cup B$ (Figure 4.16).

Also, if

$$A = \{z : -1 < \operatorname{Re} z < 1\} \quad \text{and} \quad B = \{z : |z| > 2\},$$

then A and B are connected but $A \cap B$ is not (Figure 4.17).

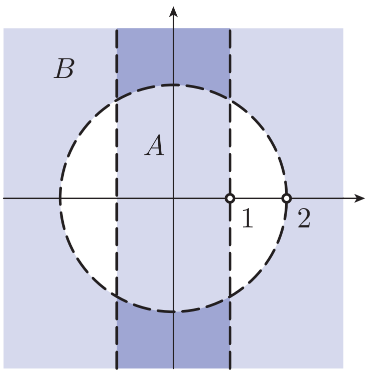


Figure 4.17 $A \cap B$ (dark shaded set) not connected

The following exercise does give one rule for obtaining 'new connected sets from old', but in practice it is usually easier to argue directly from the definition.

Exercise 4.4

Prove that if A and B are connected and $A \cap B \neq \varnothing$, then $A \cup B$ is connected.

Another general property of connected sets is given by the following result. It will be of theoretical importance later in the module.

Theorem 4.2

Let f be a continuous function whose domain A is connected. Then the image set $f(A)$ is also connected.

In words, Theorem 4.2 says that 'the continuous image of a connected set is connected'.

Proof We need to show that each pair of points in $f(A)$ can be joined by a path in $f(A)$. Such a pair of points must be of the form $f(\alpha), f(\beta)$, where $\alpha, \beta \in A$.

Because the set A is connected, we can join α to β by a path Γ in A, which is parametrised by a continuous function

$$\gamma\colon [a,b] \longrightarrow A, \quad \text{with } \gamma(a) = \alpha,\ \gamma(b) = \beta,$$

as shown on the left-hand side of Figure 4.18.

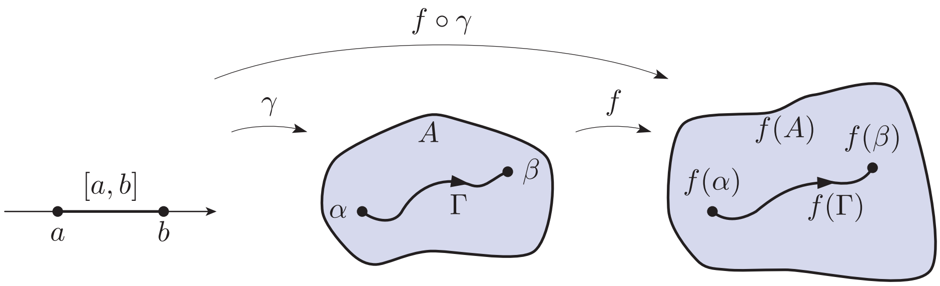


Figure 4.18 A path Γ and image path $f(\Gamma)$

Now consider the function $f \circ \gamma\colon [a,b] \longrightarrow f(A)$ shown in Figure 4.18. Then $f \circ \gamma$ is continuous, by the Composition Rule, and

$$(f \circ \gamma)(a) = f(\gamma(a)) = f(\alpha),$$
$$(f \circ \gamma)(b) = f(\gamma(b)) = f(\beta).$$

Hence $f(\alpha), f(\beta)$ are joined in $f(A)$ by the path

$$f(\Gamma) = (f \circ \gamma)([a,b]) \subseteq f(A),$$

as required. ■

Note that the continuous image of an open set need *not* be open; for example, the function $f(z) = 0$ $(z \in \mathbb{C})$ is continuous but $f(\mathbb{C}) = \{0\}$ is not open.

4.4 Regions

As noted earlier, the domains of many complex functions are both open and connected. It is useful, therefore, to introduce a name for such sets.

Definition

A **region** is a non-empty, connected, open subset of $\mathbb{C}$.

Note that the empty set $\varnothing$ is not a region.

In Subsections 4.2 and 4.3 you saw many examples of sets that are both open and connected. For future reference, we present the following list of basic types of region.

Theorem 4.3 Basic Regions

The following subsets of $\mathbb{C}$ are regions:

(a) any open disc

(b) any open half-plane

(c) the complement of any closed disc

(d) any open annulus

(e) any open rectangle

(f) any open sector (including cut planes)

(g) the set $\mathbb{C}$ itself.

We often encounter domains of functions that *are* regions but that do not fall within this basic list. In most cases it is straightforward to verify directly that a set is open and connected. However, one general result about forming regions is sometimes useful. In this result, and elsewhere, we designate a general region by the letter $\mathcal{R}$, using a special typeface to avoid confusion with other uses of the letter R.

Theorem 4.4

If $\mathcal{R}$ is a region and $\alpha_0 \in \mathcal{R}$, then $\mathcal{R} - \{\alpha_0\}$ is also a region.

Proof Since $\mathcal{R}$ is a region, it contains an open disc centred at α_0; hence $\mathcal{R} - \{\alpha_0\}$ is non-empty.

Since

$$\mathcal{R} - \{\alpha_0\} = \mathcal{R} \cap (\mathbb{C} - \{\alpha_0\}),$$

we deduce that $\mathcal{R} - \{\alpha_0\}$ is open, by Theorem 4.1 and Exercise 4.1(a).

Now suppose that $\alpha, \beta \in \mathcal{R} - \{\alpha_0\}$. Since $\alpha, \beta \in \mathcal{R}$ and $\mathcal{R}$ is a region, we can join α to β by a path Γ in $\mathcal{R}$, and this path also lies in $\mathcal{R} - \{\alpha_0\}$ if $\alpha_0 \notin \Gamma$.

If, however, Γ does contain α_0, then we choose an open disc

$$\{z : |z - \alpha_0| < r\} \subseteq \mathcal{R}$$

(which is possible since $\mathcal{R}$ is a region) and modify Γ inside this disc, in order to avoid α_0 (Figure 4.19). The resulting path joins α to β in $\mathcal{R} - \{\alpha_0\}$, so $\mathcal{R} - \{\alpha_0\}$ is connected. Therefore $\mathcal{R} - \{\alpha_0\}$ is a region. ■

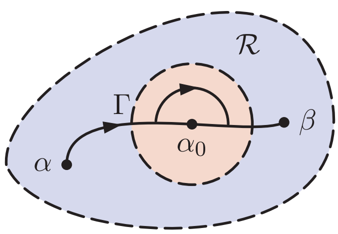


Figure 4.19 Modifying Γ to avoid α_0, by means of a circular arc

By applying Theorem 4.4 repeatedly, we deduce that if $\mathcal{R}$ is a region and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathcal{R}$, then $\mathcal{R} - \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is also a region. For example,

$$\mathcal{R} = \mathbb{C} - \{-\pi/2, \pi/2\}$$

is a region. It is tempting to think that Theorem 4.4 can be used to deduce that the domain of $f(z) = \tan z$, namely

$$\mathbb{C} - \left\{\left(n + \tfrac{1}{2}\right)\pi : n \in \mathbb{Z}\right\},$$

is a region. However, in this case we are removing from $\mathbb{C}$ an *infinite* set of points, so the result does not follow from Theorem 4.4. For this example, it is necessary to work directly from the definition of a region, which we now ask you to do.

Exercise 4.5

Prove that $\mathcal{R} = \mathbb{C} - \left\{\left(n + \tfrac{1}{2}\right)\pi : n \in \mathbb{Z}\right\}$ is a region.

Further exercises

Exercise 4.6

Sketch each of the following sets and in each case write down whether it is open, convex, connected, a region.

(a) $A = \{z : |z - i| < 2\}$ (b) $B = \{z : 1 \leq |z - 1| < 2\}$

(c) $C = \{z : \operatorname{Im} z < -1\}$ (d) $A \cup C$ (e) $B \cap C$ (f) $A - \{0\}$

Exercise 4.7

Justify your answers to Exercise 4.6(a).

Exotic regions

You should be aware that there are far more exotic regions than the basic regions such as discs, half-planes, sectors and annuli that you have met so far. Here we present a selection of striking regions, shown in Figure 4.20. Exceptionally, in illustrating these regions we do not use dashed lines for the boundaries (which are not included in the sets) but instead use solid lines. This is because dashed lines would make some of the figures look confusing.

The region in Figure 4.20(a) is constructed by removing an infinite number of vertical line segments from a rectangle, leaving small gaps alternately at the top and bottom. The line segments accumulate on the right-hand side of the rectangle (only a few of them are shown in the figure). We call this the *impossible prison* because it would take a person an infinitely long time to walk out of a prison shaped in this way that has a single door on the far right.

The impossible prison is commonly used in advanced complex analysis texts for illustrating properties of boundaries of regions.

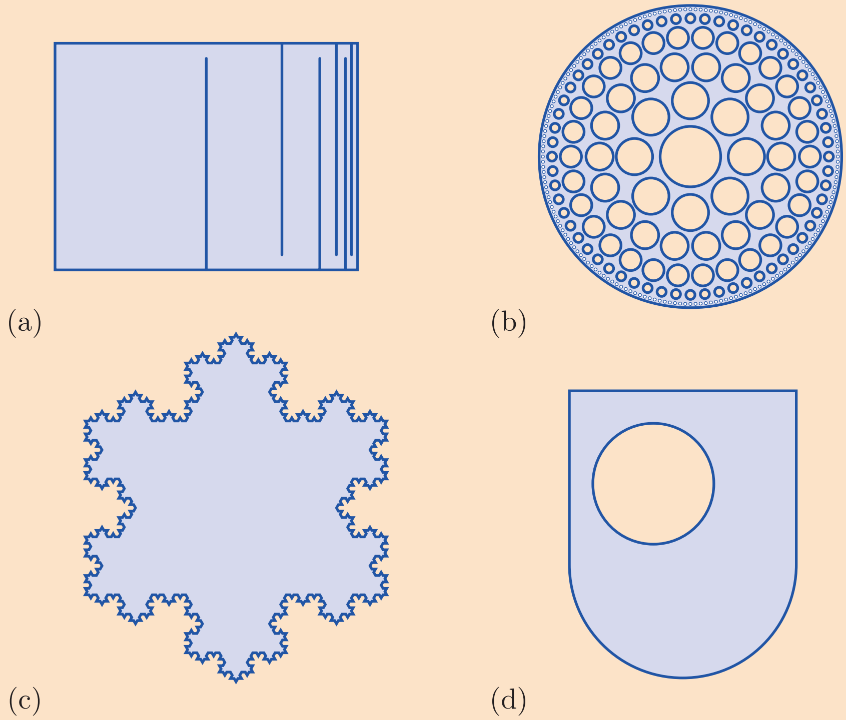


Figure 4.20 (a) Impossible prison (b) Swiss cheese (c) Koch snowflake (d) OU logo

Figure 4.20(b) is the interior of a *Swiss cheese*: a disc with infinitely many circular holes. This region also plays an important role in complex analysis, in deeper parts of the subject of conformal mappings, a subject that you will meet in Unit C3.

The region in Figure 4.20(c) is the *Koch snowflake*, which was constructed in 1904 by the Swedish mathematician Helge von Koch (1870–1924). It is a *fractal* formed from a triangle by repeatedly adding successively smaller triangles to the outside of each line segment on its boundary, infinitely many times. The first few stages in the construction of the Koch snowflake are displayed in Figure 4.21. The Koch snowflake shows that regions need not be bounded by well-behaved paths.

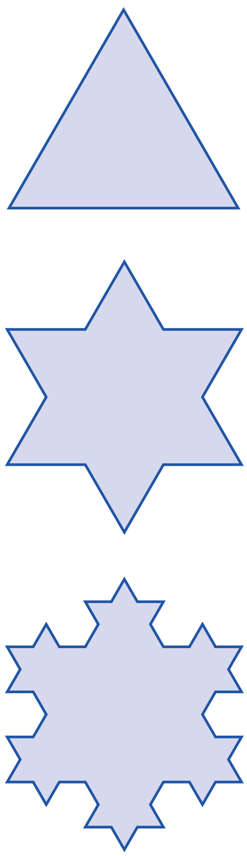


Figure 4.21 First three stages of the construction of the Koch snowflake

The last region, Figure 4.20(d), is the *OU logo*.

5 The Extreme Value Theorem

After working through this section, you should be able to:

- determine whether or not a given subset of $\mathbb{C}$ is *closed*, *bounded*
- explain the statement 'E is a compact subset of $\mathbb{C}$'
- state the Extreme Value Theorem and use its corollary, the Boundedness Theorem, to prove that certain functions are bounded on compact sets
- identify the *interior*, *exterior* and *boundary* of a given subset of $\mathbb{C}$.

5.1 Compact sets

Given a real function $f : \mathbb{R} \longrightarrow \mathbb{R}$, we often wish to determine the extreme values (that is, the absolute maximum and minimum values) of the function on a given interval I. For example, if

$$f(x) = x^2 \quad \text{and} \quad I = [-1, 2],$$

then the maximum value of f on I is $f(2) = 4$ and the minimum value of f on I is $f(0) = 0$ (Figure 5.1).

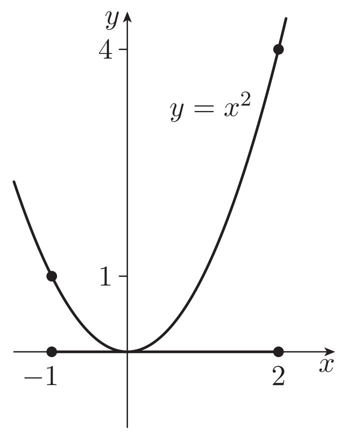


Figure 5.1 Graph of $y = x^2$

It is natural to seek a general result that states that certain types of functions *always* attain maximum and minimum values on certain types of interval. The examples in Figure 5.2 indicate that such a result cannot hold if we allow f to be discontinuous, or if we allow I to be an open interval.

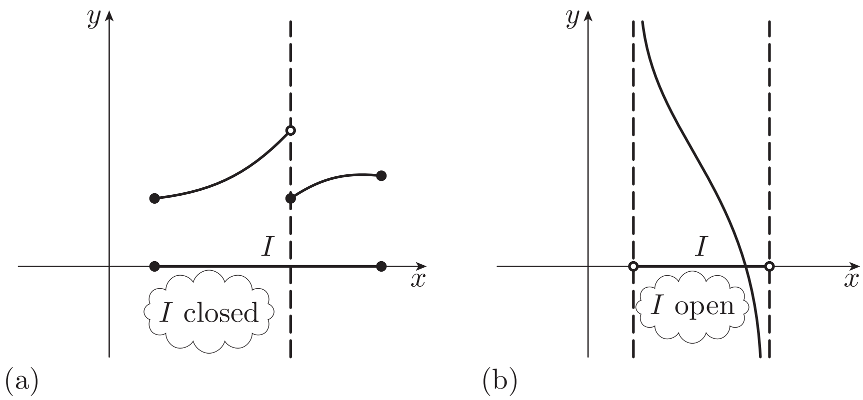


Figure 5.2 (a) f is discontinuous, and attains a minimum but not a maximum value on I (b) f is continuous, but I is open, and f attains neither a minimum nor a maximum value on I

However, if we insist that f is continuous and that $I = [a, b]$ is a closed interval with finite endpoints (in the domain of f), then f must attain maximum and minimum values on I. This result, from real analysis, is called the Extreme Value Theorem.

Our aim in this final section of the unit is to obtain a version of the Extreme Value Theorem that applies to complex functions. We have already introduced the idea of a continuous complex function in Section 2, so it remains to determine the appropriate type of subset of $\mathbb{C}$ with which to replace the closed interval $I = [a, b]$.

One good candidate for such a type of subset is a closed disc

$$\{z : |z - \alpha| \leq r\},$$

where $\alpha \in \mathbb{C}$ and $r > 0$. Such a set includes its boundary points, so the type of counterexample illustrated in Figure 5.2(b) is not possible. (The terms 'boundary point' and 'boundary', for which we have not yet required precise meanings, are defined formally near the end of this subsection.) More generally, we define a new type of set which has the property of including all its boundary points. Such sets can most neatly be defined as follows.

Definition

A set E in $\mathbb{C}$ is **closed** if its complement $\mathbb{C} - E$ is open.

To prove that a set E is closed, we must show that its complement is open.

Example 5.1

Prove that each of the following sets is closed.

(a) $\{z : \operatorname{Re} z \leq 0\}$ (b) $\{z : |z| \leq 1\}$ (c) $\{0\}$

Solution

In Example 4.1(a) and (b) and Exercise 4.1(a) we showed that the complement of each of these sets is open, as required.

More generally, any closed half-plane or closed disc is a closed set; further examples are given in the following exercise.

Exercise 5.1

Prove that each of the following sets is closed.

(a) $\{z : |z| \geq 1\}$ (b) $\{z : |z| = 1\}$

As for open sets, we can obtain 'new closed sets from old ones' by using the following Combination Rules.

Theorem 5.1 Combination Rules for Closed Sets

If E_1 and E_2 are closed sets, then so are

(a) $E_1 \cup E_2$

(b) $E_1 \cap E_2$.

Proof Since E_1 and E_2 are closed, we know that $\mathbb{C} - E_1$ and $\mathbb{C} - E_2$ are open. Next we use two properties of sets known as *De Morgan's Laws*. They can be checked by drawing Venn diagrams. These laws say that

$$\mathbb{C} - (E_1 \cup E_2) = (\mathbb{C} - E_1) \cap (\mathbb{C} - E_2)$$

and

$$\mathbb{C} - (E_1 \cap E_2) = (\mathbb{C} - E_1) \cup (\mathbb{C} - E_2),$$

and hence these sets are both open, by Theorem 4.1. Thus $E_1 \cup E_2$ and $E_1 \cap E_2$ are both closed, as required. ■

For example, by Theorem 5.1(b), the circle

$$E = \{z : |z| = 1\}$$

is closed because it is the intersection of the two closed sets

$$\{z : |z| \leq 1\} \quad \text{and} \quad \{z : |z| \geq 1\}.$$

The following corollary extends Theorem 5.1 to any finite number of sets. It can be proved by applying the Principle of Mathematical Induction to Theorem 5.1.

Corollary

If $E_1, E_2, \ldots, E_n$ are closed sets, then so are

(a) $E_1 \cup E_2 \cup \cdots \cup E_n$

(b) $E_1 \cap E_2 \cap \cdots \cap E_n$.

Exercise 5.2

Use Theorem 5.1 or its corollary to prove that each of the following sets is closed.

(a) $\{z : -1 \leq \operatorname{Re} z \leq 1,\ -1 \leq \operatorname{Im} z \leq 1\}$ (b) $\{z : \operatorname{Im} z = 0\}$

Warning! If a set contains some but not all of its boundary points, then it is *neither* open *nor* closed. For example, the set

$$A = \{z : |z| < 1\} \cup \{1\},$$

shown in Figure 5.3, is not open, because no disc $\{z : |z - 1| < r\}$ lies entirely in A, and it is not closed, because its complement is not open: no disc $\{z : |z - i| < r\}$ lies entirely in $\mathbb{C} - A$.

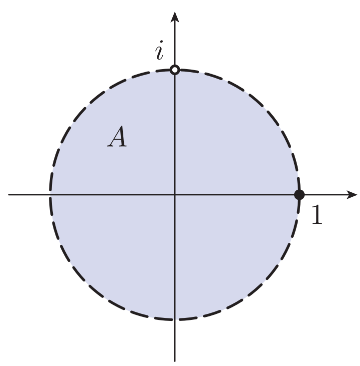


Figure 5.3 The set A is neither open nor closed

On the other hand, since the sets $\mathbb{C}$ and $\varnothing$ are both open, they are also both closed (being complementary). However, $\mathbb{C}$ and $\varnothing$ are the only subsets of $\mathbb{C}$ that are both open and closed.

The other new concept that we need is that of a *bounded* set.

Definitions

A set E in $\mathbb{C}$ is **bounded** if it is contained in some closed disc. A set is **unbounded** if it is not bounded.

For example, every open disc or closed disc is bounded, but every half-plane is unbounded.

Exercise 5.3

Determine which of the following sets are closed and which are bounded.

(a) $\{z : |z| = 1\}$ (b) $\{z : \operatorname{Im} z = 0\}$

(c) $\{z : -1 < \operatorname{Re} z < 1,\ -1 < \operatorname{Im} z < 1\}$

It turns out that the appropriate conditions for a set E to satisfy, in order that the Extreme Value Theorem holds on E, are that E is closed and bounded. It is convenient, therefore, to give such sets a name; they are called *compact sets*.

Definition

A set E in $\mathbb{C}$ is **compact** if it is closed and bounded.

We remark that this is one of several equivalent definitions of compactness for subsets of $\mathbb{C}$ discussed in texts on topology and metric spaces.

For examples of compactness, every circle is compact, every closed disc is compact, but open discs are not compact (they are not closed), and half-planes are not compact (they are not bounded).

Exercise 5.4

Prove that the set

$$\{z : -1 \le \operatorname{Re} z \le 1,\ -1 \le \operatorname{Im} z \le 1\}$$

is compact.

We are now in a position to state a version of the Extreme Value Theorem for complex functions.

The proof of this theorem is deferred to the next subsection, since it is rather involved.

Theorem 5.2 Extreme Value Theorem

Let f be a function that is continuous on a compact set E. Then there are numbers α and β in E such that

$$|f(\beta)| \le |f(z)| \le |f(\alpha)|, \quad \text{for all } z \in E.$$

Figure 5.4 shows the geometric interpretation of the Extreme Value Theorem: the image set $f(E)$ lies in the set

$$\{w : |f(\beta)| \le |w| \le |f(\alpha)|\}.$$

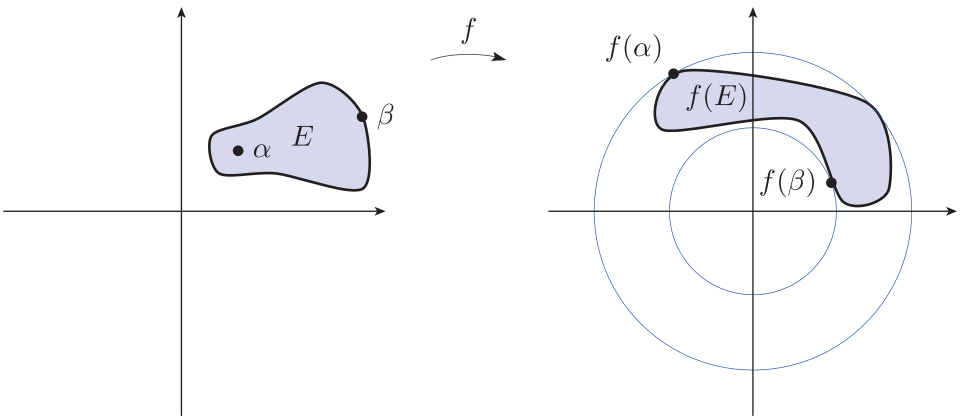


Figure 5.4 The image $f(E)$ lies between the circles centred at the origin with radii $|f(\beta)|$ and $|f(\alpha)|$

Note that the numbers α and β need not be unique, as indicated in Figure 5.4, because in this case the set $f(E)$ meets the circle of radius $|f(\alpha)|$ at two points.

It can often be extremely difficult to determine the maximum and minimum values of $|f(z)|$ on a given compact set. However, in many applications we do not need the actual maximum and minimum values. It is often enough to know that the function f is **bounded** on E; that is, the image set $f(E)$ is a bounded set. If f is continuous on E, then this follows immediately from the Extreme Value Theorem.

Theorem 5.3 Boundedness Theorem

Let f be a function that is continuous on a compact set E. Then there is a number M such that

$$|f(z)| \le M, \quad \text{for all } z \in E.$$

For a given function f and set E, we can often find an explicit value of M such that $|f(z)| \le M$, for all $z \in E$. For example, if

$$f(z) = z^3 + 2z - i \quad \text{and} \quad E = \{z : |z| \le 1\},$$

then, for all z in E,

$$\begin{aligned} |f(z)| &= |z^3 + 2z - i| \\ &\leq |z|^3 + 2|z| + 1 \qquad \text{(by the Triangle Inequality)} \\ &\leq 1 + 2 + 1 = 4 \qquad \text{(since } |z| \leq 1\text{)}. \end{aligned}$$

Therefore in this example we can take $M = 4$ so that $|f(z)| \leq M$, for all $z \in E$. However, the Boundedness Theorem can be applied even when the formula for f is complicated or unknown – as long as f is continuous.

Exercise 5.5

In each of the following cases, prove in two ways that the given function f is bounded on the given set E: by using the Boundedness Theorem *and* by explicit estimation.

(a) $f(z) = e^{1/z}, \quad E = \{z : |z| = \frac{1}{2}\}$

(b) $f(z) = \sin z, \quad E = \{z : |z| \leq 27\}$

(c) $f(z) = \dfrac{z^2 + 1}{z - 2i}, \quad E = \{z : -1 \leq \operatorname{Re} z \leq 1,\ -1 \leq \operatorname{Im} z \leq 1\}$

Figure 5.4 suggests that if E is compact and f is continuous on E, then the image set $f(E)$ is also compact. This is indeed the case.

Theorem 5.4

Let f be a function that is continuous on a compact set E. Then $f(E)$ is compact.

In words, Theorem 5.4 says that 'the continuous image of a compact set is compact'.

Proof We know, by the Boundedness Theorem, that $f(E)$ is bounded. Thus we need prove only that $f(E)$ is closed, that is, $\mathbb{C} - f(E)$ is open.

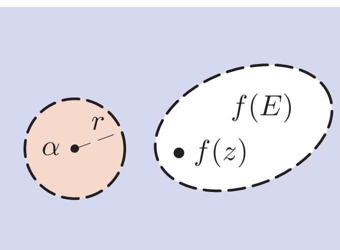


Figure 5.5 Open disc centred at α that lies in $\mathbb{C} - f(E)$

Suppose, therefore, that $\alpha \in \mathbb{C} - f(E)$. We want to find an open disc centred at α that lies entirely in $\mathbb{C} - f(E)$ (see Figure 5.5). To do this, consider the function

$$g(z) = f(z) - \alpha,$$

which is non-zero on E (since $f(z) \neq \alpha$, for $z \in E$) and continuous there. By the Extreme Value Theorem, there exists $\beta \in E$ such that

$$|g(z)| \geq |g(\beta)|, \quad \text{for all } z \in E;$$

that is,

$$|f(z) - \alpha| \geq r, \quad \text{for all } z \in E,$$

where $r = |g(\beta)|$. This number r is positive because g is non-zero on E. So the open disc with centre α and radius r lies in $\mathbb{C} - f(E)$, as required. ■

The boundary of a set

So far, we have treated the concept of the boundary of a set rather informally. It has been easy to 'see' what the boundary is; for example, the circle $\{z : |z| = 1\}$ is the boundary of the open disc $\{z : |z| < 1\}$. However, this concept can be made precise, and we now do this. (We will need this precision later in the module.) First we identify two types of point that are definitely not boundary points.

Definitions

Let A be a subset of $\mathbb{C}$ and let $\alpha \in \mathbb{C}$. Then

- α is an **interior point** of A if there is an open disc centred at α that lies entirely in A
- α is an **exterior point** of A if there is an open disc centred at α that lies entirely outside A.

The set of interior points of A forms the **interior** of A, written $\operatorname{int} A$, and the set of exterior points of A forms the **exterior** of A, written $\operatorname{ext} A$.

For example, if $A = \{z : |z| < 1\}$, then

$$\operatorname{int} A = \{z : |z| < 1\} = A \quad \text{and} \quad \operatorname{ext} A = \{z : |z| > 1\},$$

as illustrated in Figure 5.6.

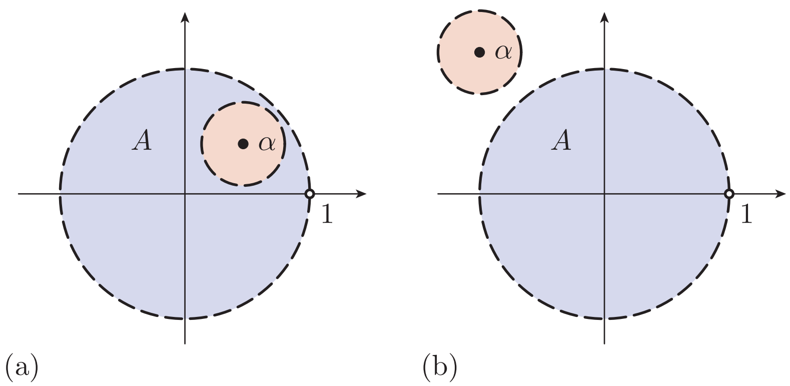


Figure 5.6 (a) $\alpha \in \operatorname{int} A$ (b) $\alpha \in \operatorname{ext} A$

If α is neither an interior point nor an exterior point of A, then each open disc centred at α must meet both A and $\mathbb{C} - A$. Points with this property are defined to be boundary points.

Definitions

Let A be a subset of $\mathbb{C}$ and let $\alpha \in \mathbb{C}$. Then α is a **boundary point** of A if each open disc centred at α contains at least one point of A and at least one point of $\mathbb{C} - A$.

The set of boundary points of A forms the **boundary** of A, written ∂A.

There is no consensus in mathematics on how the notation ∂ for a boundary should be pronounced. To describe the set ∂A, you may hear any of the phrases 'boundary A', 'dee A', 'del A', 'delta A' or 'partial A'. Given the lack of consensus, perhaps the option least likely to cause confusion is to refer to ∂A as 'the boundary of A', as is commonly done.

It follows from the definition of boundary that $\operatorname{int} A$, $\operatorname{ext} A$ and ∂A are disjoint sets, and that

$$\partial A = \mathbb{C} - (\operatorname{int} A \cup \operatorname{ext} A).$$

Thus, once you have removed $\operatorname{int} A$ and $\operatorname{ext} A$ from $\mathbb{C}$, what is left is ∂A. For example, if $A = \{z : |z| < 1\}$, then

$$\begin{aligned}\partial A &= \mathbb{C} - (\{z : |z| < 1\} \cup \{z : |z| > 1\}) \\ &= \{z : |z| = 1\},\end{aligned}$$

as expected (see Figure 5.7).

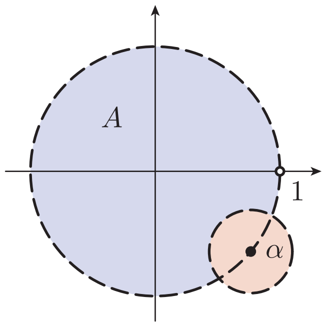


Figure 5.7 $\alpha \in \partial A$

Exercise 5.6

For each of the following sets A, write down $\operatorname{int} A$, $\operatorname{ext} A$ and ∂A.

(a) $A = \{z : |z| \leq 1\}$ (b) $A = \{x + iy : x < 0\}$ (c) $A = \mathbb{C} - \{0\}$

(d) $A = \{0\}$

You may have noticed in Exercise 5.6 that, in each case, $\operatorname{int} A$ and $\operatorname{ext} A$ are open sets, whereas ∂A is closed. This is always true.

Theorem 5.5

If A is a subset of $\mathbb{C}$, then

(a) $\operatorname{int} A$ and $\operatorname{ext} A$ are open

(b) ∂A is closed.

Proof

(a) If $\alpha \in \operatorname{int} A$, then there is an open disc $\{z : |z - \alpha| < r\}$ lying entirely in A. Since this disc is open, all points of it are interior points of A, so

$$\{z : |z - \alpha| < r\} \subseteq \operatorname{int} A.$$

Hence $\operatorname{int} A$ is open. A similar argument shows that $\operatorname{ext} A$ is open.

(b) By part (a), $\operatorname{int} A$ and $\operatorname{ext} A$ are open; hence, by Theorem 4.1, $\operatorname{int} A \cup \operatorname{ext} A$ is open. Since

$$\partial A = \mathbb{C} - (\operatorname{int} A \cup \operatorname{ext} A),$$

it follows that ∂A is closed. ■

5.2 Proof of the Extreme Value Theorem

This subsection may be omitted on a first reading.

We begin with a lemma about closed sets.

Lemma 5.1

If E is a closed set and (z_n) is a convergent sequence in E with limit α, then $\alpha \in E$.

Proof Suppose, in fact, that $\alpha \in \mathbb{C} - E$. Since $\mathbb{C} - E$ is open, it contains an open disc $\{z : |z - \alpha| < r\}$, which in turn contains all but a finite number of the terms of (z_n) (since $z_n \to \alpha$ as $n \to \infty$). However, this contradicts the fact that (z_n) is in E. We conclude that $\alpha \in E$, as required. ■

There is a converse to Lemma 5.1, which says that if the limit of each convergent sequence in the set E also lies in E, then E is closed. However, we do not need this converse result.

Next we need a fundamental theorem about nested closed rectangles (which will also be required later in the module). It is sometimes referred to as the Chinese Box Theorem. To prove this result we make use of the Monotone Convergence Theorem, a result from real analysis, which tells us that

> if the real sequence (a_n) is increasing and bounded above, then (a_n) is convergent.

(In fact, this theorem also says that if (a_n) is decreasing and bounded below, then (a_n) is convergent.)

Theorem 5.6 Nested Rectangles Theorem

Let $R_0, R_1, R_2, \ldots$ be a sequence of closed rectangles with sides parallel to the axes, and with diagonals of lengths $s_0, s_1, s_2, \ldots$, such that

$$R_0 \supseteq R_1 \supseteq R_2 \supseteq \cdots \quad \text{and} \quad \lim_{n\to\infty} s_n = 0.$$

Then there is a unique complex number α that lies in all of the rectangles R_n. Moreover, for each positive number ε, there is an integer N such that

$$R_n \subseteq \{z : |z - \alpha| < \varepsilon\}, \quad \text{for all } n > N. \tag{5.1}$$

Statement (5.1) of the theorem says, roughly speaking, that the sequence of rectangles (R_n) converges to the point α as $n \to \infty$, as indicated in Figure 5.8.

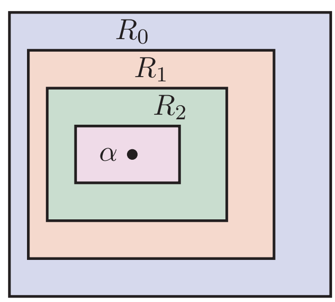


Figure 5.8 Nested rectangles

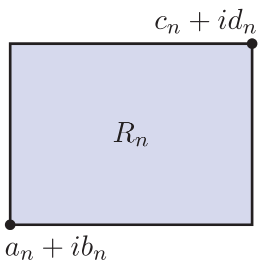


Figure 5.9 The rectangle R_n

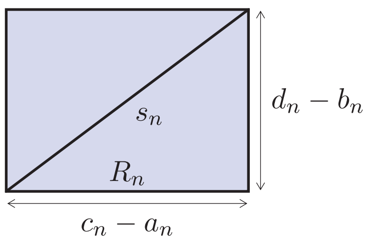


Figure 5.10 A diagonal of length s_n

Proof Let

$$R_n = \{x + iy : a_n \le x \le c_n,\ b_n \le y \le d_n\}, \quad n = 0, 1, 2, \ldots,$$

as shown in Figure 5.9, so

$$\begin{aligned} &a_0 \le a_1 \le a_2 \le \cdots \le c_2 \le c_1 \le c_0, \\ &b_0 \le b_1 \le b_2 \le \cdots \le d_2 \le d_1 \le d_0. \end{aligned} \tag{5.2}$$

Thus the sequence (a_n) is increasing and bounded above, by c_0, and so $\lim\limits_{n\to\infty} a_n$ exists by the Monotone Convergence Theorem from real analysis.

Let $a = \lim\limits_{n\to\infty} a_n$. Also, as shown in Figure 5.10,

$$0 \le c_n - a_n \le s_n, \quad \text{for } n = 0, 1, 2, \ldots,$$

so $\lim\limits_{n\to\infty} (c_n - a_n) = 0$, by the Squeeze Rule for sequences. Thus

$$\begin{aligned} \lim_{n\to\infty} c_n &= \lim_{n\to\infty} (a_n + (c_n - a_n)) \\ &= \lim_{n\to\infty} a_n + \lim_{n\to\infty} (c_n - a_n) \\ &= a + 0 = a. \end{aligned}$$

Likewise,

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} b_n = b,$$

for some real number b. Since, by (5.2),

$$\begin{aligned} &a_n \le a \le c_n, \quad \text{for } n = 0, 1, 2, \ldots, \\ &b_n \le b \le d_n, \quad \text{for } n = 0, 1, 2, \ldots, \end{aligned}$$

it follows that $\alpha = a + ib \in R_n$, for all n.

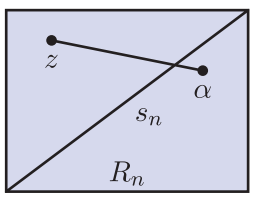


Figure 5.11 $|z - \alpha| \le s_n$

Now if $z \in R_n$, then

$$|z - \alpha| \le s_n,$$

as indicated in Figure 5.11.

Since $s_n \to 0$ as $n \to \infty$, there is an integer N such that

$$s_n < \varepsilon, \quad \text{for all } n > N;$$

so

$$|z - \alpha| < \varepsilon, \quad \text{for all } n > N \text{ and } z \in R_n.$$

Thus statement (5.1) follows.

To prove that α is the only point lying in all the rectangles R_n, suppose that β also has this property. Then statement (5.1) tells us that for each positive number ε, we have $|\beta - \alpha| < \varepsilon$. If $\beta \ne \alpha$, then we can choose $\varepsilon = |\beta - \alpha|$ to obtain a contradiction. Hence $\beta = \alpha$, so α is the only point in all of the rectangles R_n. ■

We are now in a position to prove the Extreme Value Theorem.

Theorem 5.2 Extreme Value Theorem

Let f be a function that is continuous on a compact set E. Then there are numbers α and β in E such that

$$|f(\beta)| \leq |f(z)| \leq |f(\alpha)|, \quad \text{for all } z \in E.$$

Proof First we prove that there is a number α in E such that

$$|f(z)| \leq |f(\alpha)|, \quad \text{for all } z \in E.$$

Since E is bounded, we can choose a closed rectangle R_0, with diagonal length s_0 say, such that $E \subseteq R_0$. This rectangle R_0 can then be expressed as the union of four closed rectangles T_1, T_2, T_3 and T_4, each with diagonal length $\frac{1}{2}s_0$, using vertical and horizontal lines to bisect the sides of R_0 (Figure 5.12).

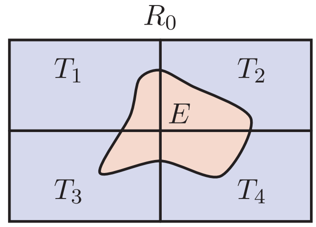


Figure 5.12 Rectangle R_0 split into four

We claim that at least one of the rectangles, T_j say, has the property that for each $z \in E$, there is some $w \in E \cap T_j$ such that

$$|f(z)| \leq |f(w)|.$$

Indeed, if the rectangle T_k does not have this property, for $k = 1, 2, 3, 4$, then there is a number z_k in E such that

$$|f(z_k)| > |f(w)|, \quad \text{for all } w \in E \cap T_k.$$

Thus if the claim is false, then such a number z_k exists for each $k = 1, 2, 3, 4$. It follows that

$$\max\{|f(z_1)|, |f(z_2)|, |f(z_3)|, |f(z_4)|\} > |f(w)|, \quad \text{for all } w \in E,$$

which is evidently a contradiction. Hence the claim is true.

Now put $R_1 = T_j$ and repeat the process with $E \cap R_1$ instead of E.

Continuing indefinitely, we obtain a sequence R_n, $n = 0, 1, 2, \ldots$, of closed rectangles such that for $n = 0, 1, 2, \ldots$:

- $R_{n+1} \subseteq R_n$
- R_n has diagonal length $\left(\frac{1}{2}\right)^n s_0$
- for each $z \in E \cap R_n$ there is some w in $E \cap R_{n+1}$ such that

$$|f(z)| \leq |f(w)|.$$

By the Nested Rectangles Theorem we deduce that there is a unique point α lying in all the rectangles R_n. Moreover, for each positive number ε, there is an integer N such that

$$R_n \subseteq \{z : |z - \alpha| < \varepsilon\}, \quad \text{for all } n > N. \tag{5.3}$$

We claim that the number α lies in E and also that

$$|f(z)| \leq |f(\alpha)|, \quad \text{for all } z \in E,$$

as required.

To prove this claim, we take *any* number z_0 in E and, observing that $E = E \cap R_0$ because $E \subseteq R_0$, we can use property (c) above to choose a sequence (z_n) such that

$$z_n \in E \cap R_n, \quad \text{for } n = 0, 1, 2, \ldots,$$

and

$$|f(z_n)| \leq |f(z_{n+1})|, \quad \text{for } n = 0, 1, 2, \ldots. \tag{5.4}$$

It follows from statement (5.3), since $z_n \in R_n$ for each n, and $R_n \subseteq R_N$ for all $n \geq N$, that $z_n \to \alpha$ as $n \to \infty$; thus $\alpha \in E$ by Lemma 5.1, since E is closed. Furthermore, since f is continuous on E, it is continuous at α, so we deduce that

$$\lim_{n\to\infty} |f(z_n)| = |f(\alpha)|.$$

Hence

$$|f(z_0)| \leq |f(\alpha)|,$$

by statement (5.4). Since z_0 was any point of E, this proves one part of the theorem.

It remains to prove that there is a number β in E such that

$$|f(\beta)| \leq |f(z)|, \quad \text{for all } z \in E.$$

If there is a point w in E such that $f(w) = 0$, then we can simply choose $\beta = w$. If there is no such number w, then $f(z) \neq 0$, for all $z \in E$, so, by the Quotient Rule for continuous functions, the function $g(z) = 1/f(z)$ is continuous on E. We can now apply the argument from the first part of this proof (to g, rather than f) to deduce the existence of a point β in E such that $|g(z)| \leq |g(\beta)|$, for all $z \in E$. But $|g(z)| = 1/|f(z)|$, so

$$|f(\beta)| \leq |f(z)|, \quad \text{for all } z \in E,$$

as required. ∎

Although it is easy to state, the Extreme Value Theorem is surprisingly tricky to prove and you should not be dispirited if you found the proof difficult. The central idea of the proof – namely, the repeated splitting of the rectangles into smaller ones – will appear again when we prove Cauchy's Theorem, the 'main result' in complex analysis.

Further exercises

Exercise 5.7

Determine which of the following sets (from Exercise 4.6) are bounded.

(a) $A = \{z : |z - i| < 2\}$ (b) $B = \{z : 1 \leq |z - 1| < 2\}$

(c) $C = \{z : \operatorname{Im} z < -1\}$ (d) $A \cup C$ (e) $B \cap C$ (f) $A - \{0\}$

Exercise 5.8

Determine whether each of the following sets is closed.

(a) $\{z : |z - i| \geq 2\}$

(b) $\{z : |z - 1| < 1 \text{ or } |z - 1| \geq 2\}$

(c) $\{z : \operatorname{Im} z \leq -1\}$

Exercise 5.9

In each of the following cases, *either* use the Boundedness Theorem to prove that the given function f is bounded on the set E, *or* explain why the Boundedness Theorem does not apply.

(a) $f(z) = \sinh z, \quad E = \{z : |z| \leq 1\}$

(b) $f(z) = \operatorname{Log} z, \quad E = \{z : |z| \leq 1\}$

(c) $f(z) = \cos z, \quad E = \{z : \operatorname{Re} z \geq 1\}$

(d) $f(z) = \dfrac{1}{z}, \quad E = \{z : 1 \leq |z| \leq 2\}$

(e) $f(z) = \dfrac{1}{z}, \quad E = \{z : 0 < |z| \leq 2\}$

(f) $f(z) = \dfrac{1}{z}, \quad E = \{z : |z| \geq 1\}$

Exercise 5.10

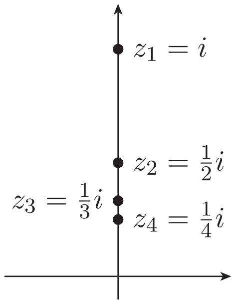
Write down the interior, exterior and boundary of the following set:

$$A = \{z : |z - 1| < 1 \text{ or } |z - 1| \geq 2\}.$$

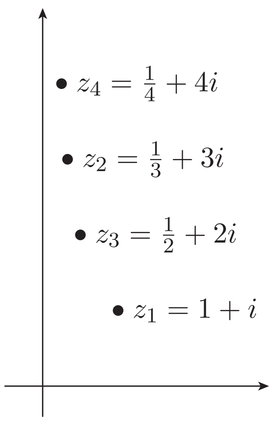
Solutions to exercises

Solution to Exercise 1.1

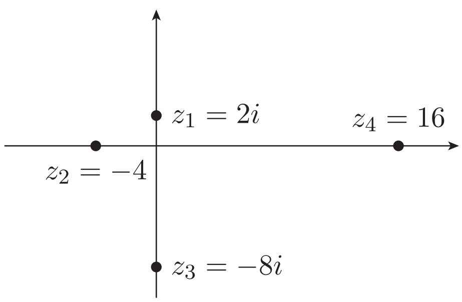
(a) $z_n = i/n, \quad n = 1, 2, \ldots$



(b) $z_n = 1/n + in, \quad n = 1, 2, \ldots$



(c) $z_n = (2i)^n, \quad n = 1, 2, \ldots$



Solution to Exercise 1.2

(a) We need to show that for each positive number ε, there is an integer N such that

$$\left|\frac{1}{\sqrt{n}}\right| = \frac{1}{\sqrt{n}} < \varepsilon, \quad \text{for all } n > N. \tag{S1}$$

But we know that

$$\frac{1}{\sqrt{n}} < \varepsilon \iff \sqrt{n} > \frac{1}{\varepsilon} \iff n > \frac{1}{\varepsilon^2}.$$

Therefore statement (S1) holds if we choose N to be any integer greater than $1/\varepsilon^2$. So $z_n = 1/\sqrt{n}$, $n = 1, 2, \ldots$, is a null sequence.

(b) We need to show that for each positive number ε, there is an integer N such that

$$\left|\frac{1+i}{n}\right| < \varepsilon, \quad \text{for all } n > N. \tag{S2}$$

But we know that $|1 + i| = \sqrt{2}$ and so

$$\left|\frac{1+i}{n}\right| < \varepsilon \iff \frac{\sqrt{2}}{n} < \varepsilon \iff n > \frac{\sqrt{2}}{\varepsilon}.$$

Therefore statement (S2) holds if we choose N to be any integer greater than $\sqrt{2}/\varepsilon$. So $z_n = (1 + i)/n$, $n = 1, 2, \ldots$, is a null sequence.

Solution to Exercise 1.3

(a) First note that

$$|0.6 + 0.8i| = ((0.6)^2 + (0.8)^2)^{1/2} = 1.$$

Hence

$$|z_n| = \left|\frac{(0.6 + 0.8i)^n}{n^2 + n}\right| = \frac{1}{n^2 + n} \leq \frac{1}{n},$$

for $n = 1, 2, \ldots$. Since $(1/n)$ is a null sequence, we deduce by the Squeeze Rule that (z_n) is null also.

(b) Since $2^n \geq n$, for $n = 1, 2, \ldots$,

$$\left|\left(\frac{i}{2}\right)^n\right| = \frac{1}{2^n} \leq \frac{1}{n}, \quad \text{for } n = 1, 2, \ldots.$$

Since $(1/n)$ is a null sequence, we deduce by the Squeeze Rule that (z_n) is null also.

Solution to Exercise 1.4

(a) The dominant term is n^3, so we divide the numerator and denominator of z_n by n^3:

$$z_n = \frac{n^3 + 2in^2 + 3}{in^3 + 1} = \frac{1 + 2i/n + 3/n^3}{i + 1/n^3}.$$

Since $(1/n)$ and $(1/n^3)$ are basic null sequences,

$$\lim_{n\to\infty} z_n = \frac{1+0+0}{i+0} = -i,$$

by the Combination Rules.

(b) Since $|3+i| = \sqrt{10}$, $|2+2i| = \sqrt{8}$ and $|1+2i| = \sqrt{5}$, the dominant term is $(3+i)^n$, so we divide the numerator and denominator of z_n by $(3+i)^n$:

$$\begin{aligned} z_n &= \frac{(3+i)^n + (2+2i)^n}{(1+2i)^n + 2(3+i)^n} \\ &= \frac{1 + ((2+2i)/(3+i))^n}{((1+2i)/(3+i))^n + 2}. \end{aligned}$$

Since

$$\begin{aligned} |(2+2i)/(3+i)| &= \sqrt{8}/\sqrt{10} < 1, \\ |(1+2i)/(3+i)| &= \sqrt{5}/\sqrt{10} < 1, \end{aligned}$$

we see that both

$$\left(\frac{2+2i}{3+i}\right)^n \quad \text{and} \quad \left(\frac{1+2i}{3+i}\right)^n,$$

$n = 1, 2, \ldots$, are basic null sequences. Hence

$$\lim_{n\to\infty} z_n = \frac{1+0}{0+2} = \frac{1}{2},$$

by the Combination Rules.

Solution to Exercise 1.5

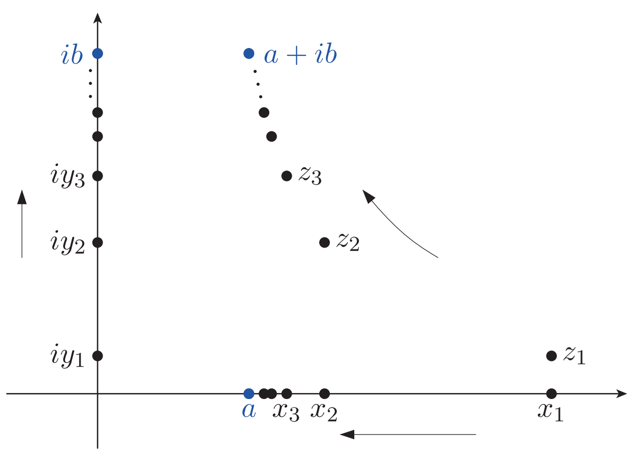
Since

$$\lim_{n\to\infty} x_n = a \quad \text{and} \quad \lim_{n\to\infty} y_n = b,$$

it follows from the Combination Rules that

$$\begin{aligned} \lim_{n\to\infty} (x_n + iy_n) &= \left(\lim_{n\to\infty} x_n\right) + i\left(\lim_{n\to\infty} y_n\right) \\ &= a + ib, \end{aligned}$$

as required. In the diagram, $z_n = x_n + iy_n$.



Solution to Exercise 1.6

We have

$$\begin{aligned} \frac{1}{z_n} &= \frac{1}{n^3 - in^2 + (1+i)n} \\ &= \frac{1/n^3}{1 - i/n + (1+i)/n^2}, \end{aligned}$$

for $n = 1, 2, \ldots$. Since $(1/n)$, $(1/n^2)$ and $(1/n^3)$ are basic null sequences, we deduce that

$$\lim_{n\to\infty} \frac{1}{z_n} = \frac{0}{1-0+0} = 0,$$

by the Combination Rules. Hence the sequence (z_n) tends to infinity, by the Reciprocal Rule.

Solution to Exercise 1.7

(a) $\frac{2i}{3}, \frac{4i}{5}, \frac{6i}{7}, \frac{8i}{9}$

(b) $\frac{3i}{4}, \frac{7i}{8}, \frac{11i}{12}, \frac{15i}{16}$

(c) $\frac{i}{2}, \frac{4i}{5}, \frac{9i}{10}, \frac{16i}{17}$

Solution to Exercise 1.8

(a) The first few terms of the sequence are

$$i, -1, -i, 1, i, -1, -i, 1, \ldots.$$

Thus (z_n) seems to have four convergent subsequences with different limits. In particular, for $k = 1, 2, \ldots$,

$$\begin{aligned} i^{4k} &= \left(i^4\right)^k = 1^k = 1 \quad \text{and} \\ i^{4k+1} &= i^{4k} i = i, \end{aligned}$$

so if $z_n = i^n$, then

$$\lim_{k\to\infty} z_{4k} = 1 \quad \text{and} \quad \lim_{k\to\infty} z_{4k+1} = i.$$

Hence, by the First Subsequence Rule, the sequence (z_n) is divergent.

(b) The first few terms of the sequence are

$$\frac{\sqrt{3}}{2}, \frac{4\sqrt{3}}{2}, 0, \frac{-16\sqrt{3}}{2}, \frac{-25\sqrt{3}}{2}, 0, \frac{49\sqrt{3}}{2}, \ldots.$$

Now $\sin(n\pi/3) = \sqrt{3}/2$ whenever $n\pi/3$ is of the form $2k\pi + \pi/3$, $k = 1, 2, \ldots$, that is, whenever $n = 6k + 1$. The subsequence (z_{6k+1}) of (z_n) satisfies

$$z_{6k+1} = (6k+1)^2 \frac{\sqrt{3}}{2}, \quad k = 1, 2, \ldots,$$

so it tends to infinity (by the Reciprocal Rule, because $1/z_{6k+1} \to 0$ as $k \to \infty$). Hence the sequence (z_n) is divergent by the Second Subsequence Rule.

Solution to Exercise 1.9

(a) To prove that the sequence (z_n) is null, we need to show that for each positive number ε, there is an integer N such that

$$\left|\frac{1+i}{2n^2-1}\right| < \varepsilon, \quad \text{for all } n > N. \tag{S3}$$

Now

$$\left|\frac{1+i}{2n^2-1}\right| = \frac{|1+i|}{2n^2-1} = \frac{\sqrt{2}}{2n^2-1},$$

for $n = 1, 2, \dots$, so

$$\begin{aligned}\left|\frac{1+i}{2n^2-1}\right| < \varepsilon &\iff \frac{\sqrt{2}}{2n^2-1} < \varepsilon \\ &\iff n > \sqrt{\tfrac{1}{2}(\sqrt{2}/\varepsilon + 1)}.\end{aligned}$$

Therefore statement (S3) holds if we choose N to be any integer greater than $\sqrt{\tfrac{1}{2}(\sqrt{2}/\varepsilon + 1)}$. So (z_n) is a null sequence.

(b) We have

$$\begin{aligned}|z_n| &= \left|\frac{1+i}{2n^2-1}\right| \\ &= \frac{\sqrt{2}}{2n^2-1} \\ &\leq \frac{\sqrt{2}}{n^2}, \quad \text{for } n = 1, 2, \dots,\end{aligned}$$

because $2n^2 - 1 \geq n^2$ for $n = 1, 2, \dots$.

Since $(1/n^2)$ is a basic null sequence, we see from the Multiple Rule that $a_n = \sqrt{2}/n^2$, $n = 1, 2, \dots$, is also a null sequence, with positive terms. It follows from the Squeeze Rule that (z_n) is a null sequence.

Solution to Exercise 1.10

(a) (z_n) is a null sequence, by Theorem 1.2(b), since it is of the form (α^n) where

$$|\alpha| = \left|\frac{1}{2} + \frac{i}{2}\right| = \frac{1}{\sqrt{2}} < 1.$$

(b) (z_n) is *not* a null sequence. In fact, using Lemma 1.1, we see that it has limit $\frac{1}{2}$ because the sequence

$$\left(\frac{1}{2} + \left(\frac{i}{2}\right)^n\right) - \frac{1}{2} = \left(\frac{i}{2}\right)^n, \quad n = 1, 2, \dots,$$

is null, by Theorem 1.2(b).

(c) (z_n) is *not* a null sequence. By Theorem 1.7(a), it tends to infinity, since it is of the form (α^n), where

$$|\alpha| = |1+i| = \sqrt{2} > 1.$$

Solution to Exercise 1.11

(a) Since $(1/n)$ is a basic null sequence,

$$\begin{aligned}\lim_{n\to\infty}\left(5 + \frac{i}{2n}\right) &= \lim_{n\to\infty} 5 + \frac{i}{2}\lim_{n\to\infty}\frac{1}{n} \\ &= 5 + 0 = 5,\end{aligned}$$

by the Sum and Multiple Rules.

(b) The dominant term in $z_n = (2n - i)/n^2$ is n^2, so we divide the numerator and denominator by n^2 to obtain

$$\begin{aligned}z_n &= \frac{2n-i}{n^2} \\ &= \frac{2/n - i/n^2}{1} \\ &= \frac{2}{n} - \frac{i}{n^2}.\end{aligned}$$

Since $(1/n)$ and $(1/n^2)$ are basic null sequences,

$$\lim_{n\to\infty} z_n = (2 \times 0) - (i \times 0) = 0,$$

by the Sum and Multiple Rules.

(c) The dominant term in $z_n = (n-i)/(n+i)$ is n, so we divide the numerator and denominator by n to obtain

$$\begin{aligned}z_n &= \frac{n-i}{n+i} \\ &= \frac{1 - i/n}{1 + i/n}.\end{aligned}$$

Since $(1/n)$ is a basic null sequence,

$$\lim_{n\to\infty} z_n = \frac{1-0}{1+0} = 1,$$

by the Combination Rules.

(d) The dominant term in

$$z_n = (n^3 + 3in - 2)/(4n^3 - in^2)$$

is n^3, so we divide the numerator and denominator by n^3 to obtain

$$\begin{aligned} z_n &= \frac{n^3 + 3in - 2}{4n^3 - in^2} \\ &= \frac{1 + 3i/n^2 - 2/n^3}{4 - i/n}. \end{aligned}$$

Since $(1/n)$, $(1/n^2)$ and $(1/n^3)$ are basic null sequences,

$$\lim_{n\to\infty} z_n = \frac{1 + 3i \times 0 - 2 \times 0}{4 - i \times 0} = \frac{1}{4},$$

by the Combination Rules.

(e) Since $|1+i| = \sqrt{2}$, $|\sqrt{3} - i| = 2$ and $|2 - 2i| = \sqrt{8}$, the dominant term in

$$z_n = \frac{(1+i)^n + (\sqrt{3} - i)^n}{3(2-2i)^n - 1}$$

is $(2-2i)^n$, so we divide the numerator and denominator by $(2-2i)^n$. Thus

$$z_n = \frac{\left(\dfrac{1+i}{2-2i}\right)^n + \left(\dfrac{\sqrt{3}-i}{2-2i}\right)^n}{3 - 1/(2-2i)^n}.$$

Let

$$\alpha_1 = \frac{1+i}{2-2i}, \quad \alpha_2 = \frac{\sqrt{3}-i}{2-2i}, \quad \alpha_3 = \frac{1}{2-2i}.$$

Then

$$\begin{aligned} |\alpha_1| &= \sqrt{2}/\sqrt{8} = 1/2 < 1, \\ |\alpha_2| &= 2/\sqrt{8} = 1/\sqrt{2} < 1, \\ |\alpha_3| &= 1/\sqrt{8} < 1. \end{aligned}$$

Hence (α_1^n), (α_2^n) and (α_3^n) are basic null sequences, by Theorem 1.2(b), so

$$\lim_{n\to\infty} z_n = \frac{0+0}{3-0} = 0,$$

by the Combination Rules.

Solution to Exercise 1.12

(a) Observe that $1/z_n = i/n$. Since (i/n) is a null sequence (by Example 1.1), the sequence (n/i) tends to infinity, by the Reciprocal Rule.

(b) Observe that

$$|z_n| = |e^{in}| = 1, \quad \text{for } n = 1, 2, \ldots.$$

Thus, for example, there is no integer N such that

$$|z_n| > 2, \quad \text{for all } n > N;$$

hence the definition of a sequence tending to infinity does not hold with $M = 2$, so the sequence (z_n) does not tend to infinity.

(c) Observe that

$$\frac{1}{z_n} = \frac{(1+i)^n}{(\sqrt{3}-i)^n - 1}.$$

Since $|1+i| = \sqrt{2}$ and $|\sqrt{3} - i| = 2$, the dominant term in $1/z_n$ is $(\sqrt{3}-i)^n$, so we divide the numerator and denominator by $(\sqrt{3}-i)^n$. Thus

$$\begin{aligned} \frac{1}{z_n} &= \frac{(1+i)^n}{(\sqrt{3}-i)^n - 1} \\ &= \frac{((1+i)/(\sqrt{3}-i))^n}{1 - 1/(\sqrt{3}-i)^n}. \end{aligned}$$

Let

$$\alpha_1 = \frac{1+i}{\sqrt{3}-i} \quad \text{and} \quad \alpha_2 = \frac{1}{\sqrt{3}-i}.$$

Then

$$\begin{aligned} |\alpha_1| &= \sqrt{2}/2 < 1, \\ |\alpha_2| &= 1/2 < 1. \end{aligned}$$

Hence the sequences (α_1^n) and (α_2^n) are null, by Theorem 1.2(b), so

$$\lim_{n\to\infty} \frac{1}{z_n} = \frac{0}{1-0} = 0,$$

by the Combination Rules.

Hence $(1/z_n)$ is a null sequence, so the sequence (z_n) tends to infinity, by the Reciprocal Rule.

Solution to Exercise 1.13

(a) Since $|i - 1| = \sqrt{2} > 1$, the sequence (z_n) is divergent, by Theorem 1.7(a).

(b) Observe that $z_n = \alpha^n$, where $\alpha = e^{\pi i} = -1$. Then $|\alpha| = 1$ but $\alpha \neq 1$, so we can apply Theorem 1.7(b) to see that (z_n) is divergent.

(c) The subsequence (z_{4k}) is

$$z_{4k} = 4k\cos(4k\pi i^{4k}), \quad k = 1, 2, \ldots.$$

Hence

$$z_{4k} = 4k\cos(4k\pi) = 4k,$$

so (z_{4k}) tends to infinity (by the Reciprocal Rule, because $1/(4k) \to 0$ as $k \to \infty$). Hence the sequence (z_n) is divergent, by the Second Subsequence Rule.

Solution to Exercise 1.14

To prove that

$$\lim_{n\to\infty} z_n = \alpha,$$

we must show that for each positive number ε, there is an integer N such that

$$|z_n - \alpha| < \varepsilon, \quad \text{for all } n > N. \tag{S4}$$

We know that (z_n) can be separated into two subsequences (z_{m_k}) and (z_{n_k}), each of which converges to α. Hence there are integers K_1 and K_2 such that

$$|z_{m_k} - \alpha| < \varepsilon, \quad \text{for all } k > K_1, \tag{S5}$$

and

$$|z_{n_k} - \alpha| < \varepsilon, \quad \text{for all } k > K_2. \tag{S6}$$

From statements (S5) and (S6), it follows that if we choose $N = \max\{m_{K_1}, n_{K_2}\}$, then statement (S4) is satisfied. Hence

$$\lim_{n\to\infty} z_n = \alpha.$$

Solution to Exercise 2.1

If $\lim_{n\to\infty} z_n = i$, then, by the Combination Rules for sequences,

$$\begin{aligned}\lim_{n\to\infty} (z_n^2 + 3z_n) &= \left(\lim_{n\to\infty} z_n\right)^2 + 3\lim_{n\to\infty} z_n \\ &= i^2 + 3i \\ &= -1 + 3i.\end{aligned}$$

Solution to Exercise 2.2

(a) To prove that the function $f(z) = 1$ is continuous at each $\alpha \in \mathbb{C}$, we must show that

$$z_n \to \alpha \implies f(z_n) \to f(\alpha). \tag{S7}$$

But $f(z_n) = 1$, for $n = 1, 2, \ldots$, and $f(\alpha) = 1$, so statement (S7) holds. Hence f is continuous.

(b) To prove that the function $f(z) = z$ is continuous at each $\alpha \in \mathbb{C}$, we must show that

$$z_n \to \alpha \implies f(z_n) \to f(\alpha). \tag{S8}$$

But $f(z_n) = z_n$, for $n = 1, 2, \ldots$, and $f(\alpha) = \alpha$, so statement (S8) holds. Hence f is continuous.

(c) To prove that the function $f(z) = \overline{z}$ is continuous at each $\alpha \in \mathbb{C}$, we must show that

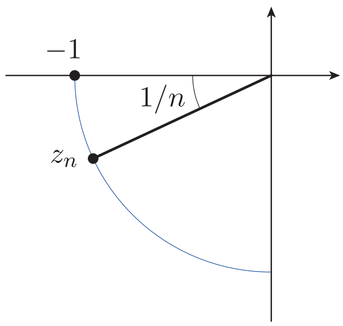
$$z_n \to \alpha \implies f(z_n) \to f(\alpha). \tag{S9}$$

But $f(z_n) = \overline{z_n}$, for $n = 1, 2, \ldots$, and $f(\alpha) = \overline{\alpha}$, so statement (S9) follows from Theorem 1.4(b). Hence f is continuous.

(d), **(e)** and **(f)**. These follow from Theorem 1.4(c), (d) and (a), respectively, as in part (c).

Solution to Exercise 2.3

If $z_n = e^{i(\pi+1/n)}$, $n = 1, 2, \ldots$, then, from the diagram, it appears that $\lim_{n\to\infty} z_n = -1$, as we show below.



Since the arc length along the unit circle $|z| = 1$ from z_n to -1 is $1/n$, and the line segment from z_n to -1 must be shorter than the arc, we have

$$|z_n - (-1)| \le \frac{1}{n}.$$

Hence $(z_n - (-1))$ is a null sequence, by the Squeeze Rule. Thus

$$\lim_{n\to\infty} z_n = -1.$$

Next, $\pi + 1/n$ is an argument of z_n, but it is not the principal argument, because it does not lie in the interval $(-\pi, \pi]$. However,

$$(\pi + 1/n) - 2\pi = -\pi + 1/n$$

does lie in the interval $(-\pi, \pi]$, so this is the principal argument of z_n; that is, $\operatorname{Arg} z_n = -\pi + 1/n$. Hence

$$\lim_{n\to\infty} \operatorname{Arg} z_n = -\pi \quad \text{(Sum Rule)}.$$

The value of $\operatorname{Arg}(-1)$ is π.

Solution to Exercise 2.4

To prove that the function $f(z) = \operatorname{Arg} z$ is discontinuous at each $\alpha \in \mathbb{R}$ with $\alpha < 0$, we must find a sequence (z_n) in $\mathbb{C} - \{0\}$, the domain of f, such that

$$z_n \to \alpha \quad \text{but} \quad f(z_n) \not\to f(\alpha).$$

Following Exercise 2.3, and the discussion after that exercise, we choose

$$z_n = |\alpha| e^{i(\pi + 1/n)}, \quad n = 1, 2, \ldots.$$

Then $z_n \to \alpha$ (by the Multiple Rule, since the sequence $(e^{i(\pi+1/n)})$ converges to -1), and

$$\lim_{n\to\infty} \operatorname{Arg} z_n = \lim_{n\to\infty} (-\pi + 1/n) = -\pi.$$

But $\operatorname{Arg} \alpha = \pi$, so $\operatorname{Arg} z_n \not\to \operatorname{Arg} \alpha$. Hence $f(z) = \operatorname{Arg} z$ is discontinuous at α.

Solution to Exercise 2.5

(a) The exponential function $g(z) = e^z$ is continuous, and so is the polynomial function $h(z) = -z^2$. Hence the function

$$f(z) = g(h(z)) = e^{-z^2}$$

is continuous, by the Composition Rule.

(b) The function

$$g(z) = \frac{z^2 + i}{z^2 - i}$$

is a rational function and hence continuous. The domain of g contains the real line (since $z^2 - i \neq 0$ for $z \in \mathbb{R}$) and hence the function

$$f(x) = \frac{x^2 + i}{x^2 - i} \quad (x \in \mathbb{R}),$$

which is the restriction of g to $\mathbb{R}$, is continuous, by the Restriction Rule.

(c) The function $g(z) = |z|$ is a continuous function, and the (real) function $h(x) = \log x$ is continuous, with domain $(0, \infty)$. Since g is continuous at each $\alpha \in \mathbb{C}$ and $g(z) > 0$ if $z \neq 0$, the function

$$f(z) = h(g(z)) = \log |z|$$

is continuous on $\mathbb{C} - \{0\}$, by the Composition Rule.

(d) The functions $g(z) = \operatorname{Re} z$, $h(z) = z^2 + 1$, $k(z) = |z|$ are all continuous. Hence

$$z \longmapsto \operatorname{Re}(z^2 + 1)$$

is continuous, by the Composition Rule, and

$$z \longmapsto |z|^2$$

is continuous, by the Product Rule.

Hence the function

$$f(z) = \operatorname{Re}(z^2 + 1) - |z|^2$$

is continuous, by the Combination Rules.

Solution to Exercise 2.6

The function $g(z) = (z^2 + 1)/(z - 2i)$ is a rational function and hence it is continuous on $\mathbb{C} - \{2i\}$.

The function $h(z) = \sin z$ is one of the basic continuous functions. Hence, by the Composition Rule, the function

$$h(g(z)) = \sin\left(\frac{z^2 + 1}{z - 2i}\right)$$

is continuous on $\mathbb{C} - \{2i\}$.

The function

$$f(z) = \sin\left(\frac{z^2 + 1}{z - 2i}\right) \quad (z \in A),$$

where $A = \{z : -1 \le \operatorname{Re} z \le 1,\ -1 \le \operatorname{Im} z \le 1\}$, is the restriction of $h \circ g$ to A, and $h \circ g$ is continuous on $\mathbb{C} - \{2i\}$; hence, by the Restriction Rule, the function f is continuous (on A).

Solution to Exercise 2.7

(a) The sequence

$$z_n = \pi - 2/n, \quad n = 1, 2, \ldots,$$

has limit π (Sum and Multiple Rules), and π is a point at which the function $f(z) = \sin z$ is continuous. Hence

$$\lim_{n\to\infty} \sin(\pi - 2/n) = \sin \pi = 0.$$

(b) The sequence

$$z_n = i + 1/n^2, \quad n = 1, 2, \ldots,$$

has limit i (Sum Rule), and i is a point at which the function $f(z) = \operatorname{Arg} z$ is continuous. Hence

$$\lim_{n\to\infty} \operatorname{Arg}(i + 1/n^2) = \operatorname{Arg} i = \pi/2.$$

(c) The sequence

$$z_n = -i\pi/2 + i/(2n), \quad n = 1, 2, \ldots,$$

has limit $-i\pi/2$ (Sum and Multiple Rules), and $-i\pi/2$ is a point at which the function $f(z) = \exp z$ is continuous. Hence

$$\lim_{n\to\infty} \exp(-i\pi/2 + i/(2n)) = \exp(-i\pi/2) = -i.$$

Solution to Exercise 2.8

(a) Let $\alpha \in \{x \in \mathbb{R} : x < 0\}$. To prove that the function $g(z) = z^{1/2}$ is discontinuous at α, we must find a sequence (z_n) in $\mathbb{C}$ (the domain of g) such that

$$z_n \to \alpha \quad \text{but} \quad g(z_n) \not\to g(\alpha).$$

Consider, for example, the sequence

$$z_n = |\alpha| e^{i(\pi+1/n)}, \quad n = 1, 2, \ldots;$$

then, as we saw in the solution to Exercise 2.4, $z_n \to \alpha$.

In terms of the principal argument,

$$z_n = |\alpha| e^{i(-\pi+1/n)},$$

so $|z_n| = |\alpha|$ and $\operatorname{Arg} z_n = -\pi + 1/n$. Thus

$$\begin{aligned} g(z_n) &= z_n^{1/2} \\ &= \exp\left(\tfrac{1}{2} \operatorname{Log} z_n\right) \\ &= \exp\left(\log |z_n|^{1/2} + i(\operatorname{Arg} z_n)/2\right) \\ &= \exp\left(\log |\alpha|^{1/2}\right) \exp(i(-\pi + 1/n)/2) \\ &= |\alpha|^{1/2} \exp(-i\pi/2 + i/(2n)). \end{aligned}$$

Now, by Exercise 2.7(c),

$$\exp(-i\pi/2 + i/(2n)) \to -i.$$

Thus, by the Multiple Rule,

$$g(z_n) \to -|\alpha|^{1/2} i.$$

But

$$\begin{aligned} g(\alpha) &= \alpha^{1/2} \\ &= \exp\left(\tfrac{1}{2} \operatorname{Log} \alpha\right) \\ &= \exp\left(\log |\alpha|^{1/2} + i(\operatorname{Arg} \alpha)/2\right) \\ &= |\alpha|^{1/2} \exp(i\pi/2) \\ &= |\alpha|^{1/2} i. \end{aligned}$$

Thus

$$z_n \to \alpha \quad \text{but} \quad g(z_n) \not\to g(\alpha),$$

so g is discontinuous at α.

(b) We now show that $g(z) = z^{1/2}$ is continuous at 0 by using the ε-δ definition of continuity. We must show that for each $\varepsilon > 0$, there is some $\delta > 0$ such that, for $z \in \mathbb{C}$,

$$|z - 0| < \delta \implies |z^{1/2} - 0^{1/2}| < \varepsilon;$$

that is,

$$|z| < \delta \implies |z^{1/2}| < \varepsilon. \tag{S10}$$

Now

$$\begin{aligned} |z^{1/2}| < \varepsilon &\iff |z^{1/2}|^2 < \varepsilon^2 \\ &\iff |z| < \varepsilon^2. \end{aligned}$$

Hence, on taking $\delta = \varepsilon^2$, statement (S10) holds, so g is continuous at 0.

Solution to Exercise 2.9

(a) The function $f(z) = z^2$ is continuous at $\alpha = 2i$. We prove this as follows.

Let (z_n) be any sequence (in $\mathbb{C}$, the domain of f) such that $z_n \to 2i$. We must show that

$$z_n \to 2i \implies f(z_n) \to f(2i) = -4. \tag{S11}$$

Now

$$\begin{aligned} f(z_n) &= z_n^2 \\ &= z_n \times z_n \\ &\to 2i \times 2i = -4, \end{aligned}$$

by the Product Rule for sequences. Hence statement (S11) holds, so $f(z) = z^2$ is continuous at $2i$.

(b) The (principal cube root) function $f(z) = z^{1/3}$ has domain $\mathbb{C}$, which includes the point -1. We prove that $f(z) = z^{1/3}$ is discontinuous at -1, by finding a sequence (z_n) such that

$$z_n \to -1 \quad \text{but} \quad f(z_n) \not\to f(-1).$$

Consider the sequence

$$z_n = e^{i(\pi+1/n)}, \quad n = 1, 2, \ldots.$$

Then $z_n \to -1$ (see Exercise 2.3, or use the continuity of $g(z) = \exp z$ at $i\pi$).

In terms of the principal argument,

$$z_n = e^{i(-\pi+1/n)},$$

so $|z_n| = 1$ and $\operatorname{Arg} z_n = -\pi + 1/n$.

Thus

$$\begin{aligned} f(z_n) &= z_n^{1/3} \\ &= \exp\left(\tfrac{1}{3}\,\mathrm{Log}\, z_n\right) \\ &= \exp(\log|z_n|^{1/3} + i(\mathrm{Arg}\, z_n)/3) \\ &= \exp(\log 1)\exp(i(-\pi + 1/n)/3) \\ &= \exp(-i\pi/3 + i/(3n)). \end{aligned}$$

Now, the sequence $(-i\pi/3 + i/(3n))$ has limit $-i\pi/3$ (by the Sum and Multiple Rules), which is a point at which the function $g(z) = \exp z$ is continuous. Hence

$$\begin{aligned} f(z_n) = \exp(-i\pi/3 + i/(3n)) &\to \exp(-i\pi/3) \\ &= \frac{1}{2} - \frac{\sqrt{3}}{2}i. \end{aligned}$$

But

$$\begin{aligned} f(-1) &= (-1)^{1/3} \\ &= \exp\left(\tfrac{1}{3}\,\mathrm{Log}(-1)\right) \\ &= \exp(\log|-1|^{1/3} + i(\mathrm{Arg}(-1))/3) \\ &= \exp(0 + i\pi/3) \\ &= \frac{1}{2} + \frac{\sqrt{3}}{2}i \\ &\neq \frac{1}{2} - \frac{\sqrt{3}}{2}i. \end{aligned}$$

Hence

$$z_n \to -1 \quad \text{but} \quad f(z_n) \not\to f(-1),$$

so f is discontinuous at -1.

Solution to Exercise 2.10

(a) The functions $g(z) = 3z^3$, $h(z) = |z|$ and $k(z) = \mathrm{Re}\, z$ are basic continuous functions. By the Product Rule,

$$z \longmapsto |z|\,\mathrm{Re}\, z$$

is continuous, and hence

$$f(z) = 3z^3 + |z|\,\mathrm{Re}\, z$$

is continuous, by the Sum Rule.

(b) The functions $g(z) = |z|$ and $h(z) = \sin z$ are basic continuous functions. Hence, by the Composition Rule, the function

$$f(z) = g(h(z)) = |\sin z|$$

is continuous.

(c) The function $g(z) = 1 + z(i-1)$ is a polynomial function and, hence, continuous. The domain of g is $\mathbb{C}$, which contains the interval $[0,1]$, and hence the function

$$f(x) = 1 + x(i-1) \quad (x \in [0,1]),$$

which is the restriction of g to $[0,1]$, is continuous, by the Restriction Rule.

(d) The function $g(z) = e^{iz}$ is continuous (by the Composition Rule) and has domain $\mathbb{C}$, which contains the interval $[0, 2\pi]$. Since $g(z) = \cos z + i \sin z$, we see that

$$f(x) = \cos x + i \sin x \quad (x \in [0, 2\pi])$$

is continuous, by the Restriction Rule.

Solution to Exercise 2.11

The domain of the function

$$f(z) = \theta,$$

where θ is the argument of z that lies in the interval $[0, 2\pi)$, is $\mathbb{C} - \{0\}$.

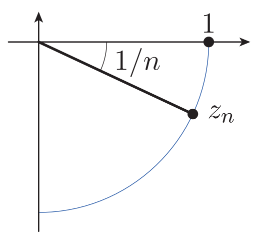
To show that f is discontinuous at 1, we find a sequence (z_n) in $\mathbb{C} - \{0\}$ such that

$$z_n \to 1 \quad \text{but} \quad f(z_n) \not\to f(1).$$

Consider the sequence

$$z_n = e^{-i/n}, \quad n = 1, 2, \ldots,$$

which satisfies $z_n \to 1$, by the continuity of the exponential function at 0.



Now

$$f(z_n) = 2\pi - \frac{1}{n}, \quad n = 1, 2, \ldots,$$

so

$$f(z_n) \to 2\pi,$$

by the Combination Rules. But $f(1) = 0$; hence $f(z_n) \not\to f(1)$, so f is discontinuous at 1.

In fact, the function f is discontinuous at each positive number.

Solution to Exercise 2.12

(a) The sequence

$$z_n = \pi + i/n, \quad n = 1, 2, \ldots,$$

has limit π, and π is a point at which the function $f(z) = \operatorname{Log} z$ is continuous. Hence

$$\begin{aligned}\lim_{n\to\infty} \operatorname{Log}(\pi + i/n) &= \operatorname{Log}\pi \\ &= \log \pi.\end{aligned}$$

(b) Consider the sequence

$$z_n = \frac{(2n+1)\pi}{2n-1}\, i, \quad n = 1, 2, \ldots.$$

The dominant term in z_n is n, so we divide the numerator and denominator by n, giving

$$z_n = \frac{(2+1/n)\pi}{2-1/n}\, i.$$

Hence, by the Combination Rules for sequences,

$$z_n \to \pi i.$$

The function $f(z) = \exp z$ is continuous at πi. Hence

$$\lim_{n\to\infty} \exp\left(\frac{(2n+1)\pi}{2n-1}\, i\right) = \exp(\pi i) = -1.$$

(c) Consider the sequence

$$z_n = \frac{(1+i)^n}{(2+i)^n}, \quad n = 1, 2, \ldots.$$

Since $|(1+i)/(2+i)| = \sqrt{2}/\sqrt{5} < 1$, (z_n) is a basic null sequence (Theorem 1.2(b)). The function $f(z) = \cos z$ is continuous at 0. Hence

$$\lim_{n\to\infty} \cos\left(\frac{(1+i)^n}{(2+i)^n}\right) = \cos 0 = 1.$$

Solution to Exercise 3.1

(a) The point $\alpha = 0$ is a limit point of $A = \{z : |z| < 1\}$ because all points of the sequence

$$z_n = \frac{1}{n+1}, \quad n = 1, 2, \ldots,$$

lie in $A - \{0\}$ and $z_n \to 0$ as $n \to \infty$.

(We remark that any null sequence in $A - \{0\}$ could be used instead of (z_n) here, and likewise the choice of sequence in each of the other parts of this exercise is by no means unique for its purpose.)

(b) The point $\alpha = i$ is a limit point of $A = \{z : \operatorname{Re} z > 0\}$ because all points of the sequence

$$z_n = \frac{1}{n} + i, \quad n = 1, 2, \ldots,$$

lie in $A - \{i\} = A$ and $z_n \to i$ as $n \to \infty$.

(c) The point $\alpha = 1$ is a limit point of $A = \{z : |z| = 1\}$ because all points of the sequence

$$z_n = e^{i/n}, \quad n = 1, 2, \ldots,$$

lie in $A - \{1\}$ and $z_n \to 1$ as $n \to \infty$ (by the continuity of the exponential function at 0).

(d) The point $\alpha = 2$ is a limit point of $A = \mathbb{C} - \{2\}$ because all points of the sequence

$$z_n = 2 + \frac{1}{n}, \quad n = 1, 2, \ldots,$$

lie in $A - \{2\}$ $(= \mathbb{C} - \{2\})$ and $z_n \to 2$ as $n \to \infty$.

(e) The point $\alpha = -1$ is a limit point of $A = \mathbb{R} - \{-1\}$ because all points of the sequence

$$z_n = -1 + \frac{1}{n}, \quad n = 1, 2, \ldots,$$

lie in $A - \{-1\}$ $(= \mathbb{R} - \{-1\})$ and $z_n \to -1$ as $n \to \infty$.

Solution to Exercise 3.2

First, note that the domain of the function

$$f(z) = \frac{z^3 + i}{z - i}$$

is $\mathbb{C} - \{i\}$ and that i is a limit point of this set. Also,

$$\begin{aligned} f(z) &= \frac{z^3+i}{z-i} \\ &= \frac{(z-i)(z^2+iz-1)}{(z-i)} \\ &= z^2 + iz - 1, \quad \text{for } z \in \mathbb{C} - \{i\}. \end{aligned}$$

Thus if (z_n) is a sequence lying in $\mathbb{C} - \{i\}$ and $z_n \to i$, then

$$\begin{aligned} f(z_n) &= z_n^2 + iz_n - 1 \\ &\to i^2 + i^2 - 1 = -3, \end{aligned}$$

by the Combination Rules for sequences, so

$$\lim_{z\to i} \frac{z^3+i}{z-i} = -3.$$

Solution to Exercise 3.3

Let $f(z) = z/(\operatorname{Re} z)$. Then the domain of f is $A = \{z : \operatorname{Re} z \neq 0\}$, and 0 is a limit point of A.

Consider the sequence

$$z_n = \frac{1}{n} + i\frac{k}{n}, \quad n = 1, 2, \ldots,$$

where k is an integer. Now, $z_n \to 0$ through $A - \{0\}$, but

$$\begin{aligned} f(z_n) &= \left(\frac{1}{n} + i\frac{k}{n}\right) \Big/ \frac{1}{n} \\ &= 1 + ik, \quad n = 1, 2, \ldots, \end{aligned}$$

so

$$\lim_{n\to\infty} f(z_n) = 1 + ik.$$

Clearly, different values of k (such as 0 and 1) lead to different limits, so $\lim_{z\to 0} f(z)$ does not exist.

Alternatively, if

$$z_n = \frac{1}{n^2} + i\frac{1}{n}, \quad n = 1, 2, \ldots,$$

then

$$\begin{aligned} f(z_n) &= \left(\frac{1}{n^2} + i\frac{1}{n}\right) \Big/ \frac{1}{n^2} \\ &= 1 + in, \quad n = 1, 2, \ldots. \end{aligned}$$

Now

$$\left|\frac{1}{1 + in}\right| \leq \frac{1}{|in|} = \frac{1}{n},$$

so $1/(1 + in) \to 0$ by the Squeeze Rule. Therefore $f(z_n) \to \infty$ as $n \to \infty$ by the Reciprocal Rule for sequences, so $\lim_{z\to 0} f(z)$ does not exist.

Solution to Exercise 3.4

(a) Observe that

$$\frac{z^2 - 4}{z - 2} = z + 2,$$

for $z \in \mathbb{C} - \{2\}$. The polynomial function $f(z) = z + 2$ is continuous on $\mathbb{C}$, and 2 is a limit point of $\mathbb{C}$. Hence, by Theorem 3.1,

$$\begin{aligned} \lim_{z\to 2} \frac{z^2 - 4}{z - 2} &= \lim_{z\to 2} f(z) \\ &= f(2) \\ &= 2 + 2 = 4. \end{aligned}$$

(b) As you saw in Exercise 3.2,

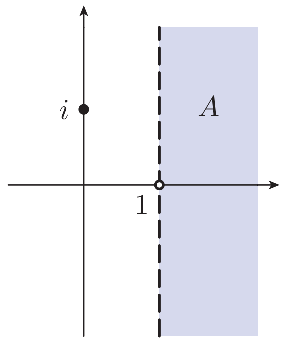
$$\frac{z^3 + i}{z - i} = z^2 + iz - 1,$$

for $z \in \mathbb{C} - \{i\}$. The polynomial function $f(z) = z^2 + iz - 1$ is continuous on $\mathbb{C}$, and i is a limit point of $\mathbb{C}$. Hence, by Theorem 3.1,

$$\begin{aligned} \lim_{z\to i} \frac{z^3 + i}{z - i} &= \lim_{z\to i} f(z) \\ &= f(i) \\ &= i^2 + (i \times i) - 1 = -3. \end{aligned}$$

Solution to Exercise 3.5

(a) The point i is not a limit point of $A = \{z : \operatorname{Re} z > 1\}$.



(b) The point 1 is a limit point of $A = \{z : \operatorname{Re} z + \operatorname{Im} z = 1\}$.

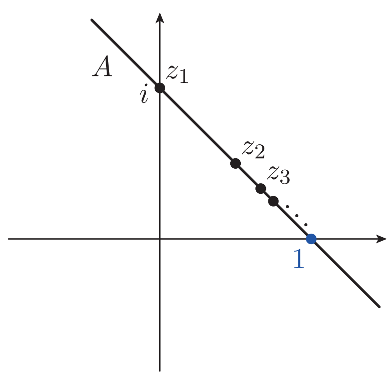
For example, the sequence

$$z_n = \left(1 - \frac{1}{n}\right) + \frac{i}{n}, \quad n = 1, 2, \ldots,$$

lies in $A - \{1\}$, and

$$\lim_{n\to\infty} z_n = 1,$$

by the Sum and Multiple Rules. Hence 1 is a limit point of A.



Solution to Exercise 3.6

(a) Observe that

$$\frac{z^3-27}{z-3} = z^2+3z+9,$$

for $z \in \mathbb{C} - \{3\}$. The polynomial function $f(z) = z^2+3z+9$ is continuous on $\mathbb{C}$, and 3 is a limit point of $\mathbb{C}$. Hence, by Theorem 3.1,

$$\begin{aligned}\lim_{z\to 3}\frac{z^3-27}{z-3} &= \lim_{z\to 3} f(z)\\ &= f(3)\\ &= 3^2+(3\times 3)+9 = 27.\end{aligned}$$

(b) Observe that

$$\frac{z^2+1}{z+i} = z-i,$$

for $z \in \mathbb{C} - \{-i\}$. The polynomial function $f(z) = z-i$ is continuous on $\mathbb{C}$, and $-i$ is a limit point of $\mathbb{C}$. Hence, by Theorem 3.1,

$$\begin{aligned}\lim_{z\to -i}\frac{z^2+1}{z+i} &= \lim_{z\to -i} f(z)\\ &= f(-i)\\ &= -i-i = -2i.\end{aligned}$$

(c) The functions $g(z) = e^z$, $h(z) = \sinh z$ and $k(z) = 1/z$ are basic continuous functions. Hence, by the Product and Sum Rules, the function

$$f(z) = e^z \sinh z + 1/z$$

is continuous (on $\mathbb{C} - \{0\}$). In particular, f is continuous at $i\pi$, a limit point of $\mathbb{C} - \{0\}$. Hence, by Theorem 3.1,

$$\begin{aligned}\lim_{z\to i\pi} f(z) &= f(i\pi)\\ &= e^{i\pi}\sinh(i\pi) + \frac{1}{i\pi}\\ &= (-1)\times(i\sin\pi) - \frac{i}{\pi} = -\frac{i}{\pi}.\end{aligned}$$

(d) It follows from Theorem 3.1 that

$$\lim_{z\to 1} \operatorname{Im} z = \operatorname{Im} 1 = 0,$$

so it looks as if $\lim_{z\to 1}(1/\operatorname{Im} z)$ does not exist.

The domain of the function $f(z) = 1/\operatorname{Im} z$ is $A = \mathbb{C} - \{z : \operatorname{Im} z = 0\}$, that is, $\mathbb{C}$ with the real axis removed. Also, 1 is a limit point of A; for example, the sequence $(1+i/n)$ lies in $A - \{1\}$, and $1+i/n \to 1$ as $n\to\infty$.

However,

$$f(1+i/n) = \frac{1}{\operatorname{Im}(1+i/n)} = n,$$

so $f(1+i/n)$ tends to infinity. Hence $\lim_{z\to 1} f(z)$ does not exist.

(e) The functions $f(z) = \operatorname{Re} z$ and $g(z) = \operatorname{Im} z$ are continuous on $\mathbb{C}$, so f/g is continuous on

$$\mathbb{C} - \{z : \operatorname{Im} z = 0\} = \mathbb{C} - \mathbb{R}.$$

Since $\operatorname{Im} i = 1 \neq 0$, we deduce that

$$\lim_{z\to i}\frac{\operatorname{Re} z}{\operatorname{Im} z} = \frac{\operatorname{Re} i}{\operatorname{Im} i} = 0,$$

by Theorem 3.1.

(f) Since $\lim_{z\to 0}\operatorname{Re} z = 0$ and $\lim_{z\to 0}\operatorname{Im} z = 0$ (by Theorem 3.1), it looks as if $\lim_{z\to 0}\dfrac{\operatorname{Re} z}{\operatorname{Im} z}$ does not exist. The function $f(z) = \dfrac{\operatorname{Re} z}{\operatorname{Im} z}$ has domain $A = \{z : \operatorname{Im} z \neq 0\}$, and 0 is a limit point of A.

Consider the sequences

$$z_n = \frac{i}{n}, \quad z_n' = \frac{1+i}{n}, \quad n = 1, 2, \ldots,$$

both of which are null sequences that lie in $A - \{0\}$. Now

$$f(z_n) = \frac{\operatorname{Re}(i/n)}{\operatorname{Im}(i/n)} = 0 \to 0,$$

whereas

$$f(z_n') = \frac{\operatorname{Re}((1+i)/n)}{\operatorname{Im}((1+i)/n)} = \frac{1/n}{1/n} = 1 \to 1,$$

as $n\to\infty$. These two limits differ, so $\lim_{z\to 0} f(z)$ does not exist.

Solution to Exercise 4.1

(a) Let $A = \mathbb{C} - \{0\}$. If $\alpha \in A$, then the open disc $\{z : |z-\alpha| < |\alpha|\}$ lies entirely in A, so A is open.

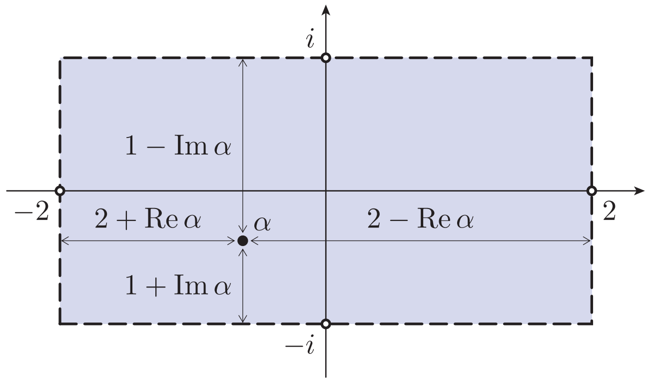
(b) Let

$$A = \{z : -2 < \operatorname{Re} z < 2,\ -1 < \operatorname{Im} z < 1\}$$

and $\alpha \in A$. Then the distance r_α from α to the boundary of A is equal to

$$\min\{2-\operatorname{Re}\alpha, 1-\operatorname{Im}\alpha, 2+\operatorname{Re}\alpha, 1+\operatorname{Im}\alpha\}.$$

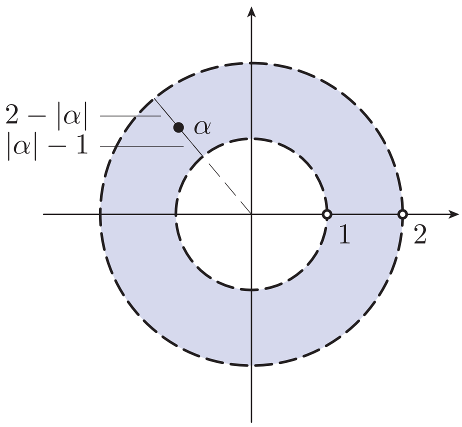
Hence $r_\alpha > 0$.



Thus the disc $\{z : |z - \alpha| < r_\alpha\}$ lies entirely in A, so A is open.

(c) Let $A = \{z : 1 < |z| < 2\}$ and $\alpha \in A$. Then the distance from α to the boundary of A is

$$r_\alpha = \min\{|\alpha| - 1, 2 - |\alpha|\} > 0.$$



Thus the disc $\{z : |z - \alpha| < r_\alpha\}$ lies entirely in A, so A is open.

(d) Let $A = \{z : \pi/3 < \operatorname{Arg} z < 2\pi/3\}$ and $\alpha \in A$. The boundary of A is

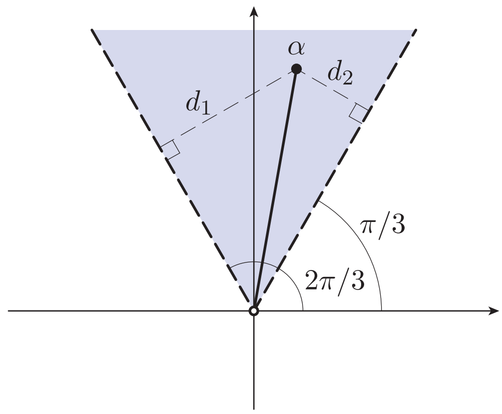
$$\{0\} \cup \{z : \operatorname{Arg} z = \pi/3\} \cup \{z : \operatorname{Arg} z = 2\pi/3\},$$

and the distance from α to the boundary of A is

$$r_\alpha = \min\{d_1, d_2\} > 0,$$

where

$$d_1 = |\alpha| \sin(2\pi/3 - \operatorname{Arg}\alpha),$$
$$d_2 = |\alpha| \sin(\operatorname{Arg}\alpha - \pi/3).$$



Thus the disc $\{z : |z - \alpha| < r_\alpha\}$ lies entirely in A, so A is open.

Solution to Exercise 4.2

(a) Each of the sets

$$A_1 = \{z : \operatorname{Re} z > 0\}, \quad A_2 = \{z : \operatorname{Im} z > 0\},$$
$$A_3 = \{z : |z| < 1\}$$

is open, so

$$\{z : \operatorname{Re} z > 0,\ \operatorname{Im} z > 0,\ |z| < 1\}$$
$$= A_1 \cap A_2 \cap A_3$$

is open, by the corollary to Theorem 4.1.

(b) Each of the sets

$$A_1 = \{z : \operatorname{Re} z > 0\}, \quad A_2 = \{z : \operatorname{Re} z < 0\}$$

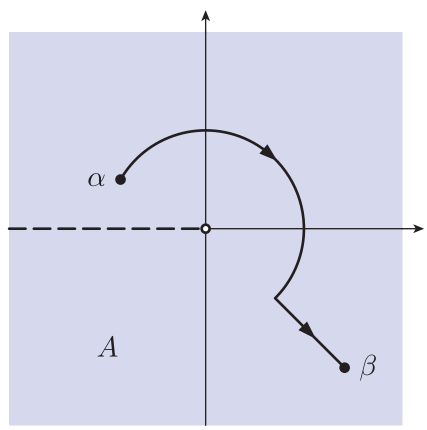
is open, so

$$\{z : \operatorname{Re} z \neq 0\} = A_1 \cup A_2$$

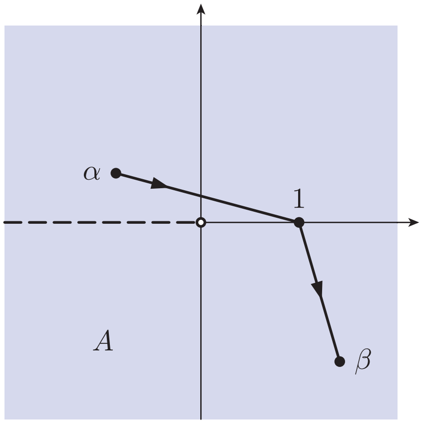
is open, by Theorem 4.1(a).

Solution to Exercise 4.3

(a) Let $A = \mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$. Then any two points α and β in A can be joined by a path in A of the type given in the solution to Example 4.2.



Alternatively, the points α and β can be joined by the path that is the union of the line segments from α to 1 and from 1 to β, both of which lie in the set A.

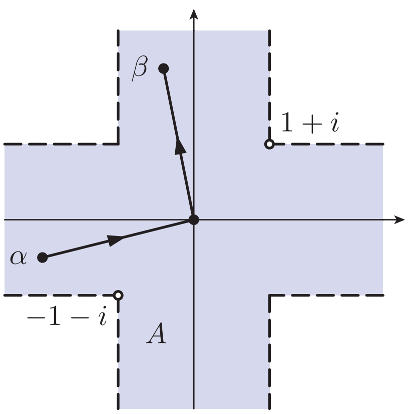


The set A is not convex because, for example, $-1+i$ and $-1-i$ cannot be joined by a line segment lying entirely in A.

(b) If α and β lie in

$$A = \{z : -1 < \operatorname{Re} z < 1 \text{ or } -1 < \operatorname{Im} z < 1\},$$

then the line segments from α to 0 and from 0 to β both lie in A. Hence α can be joined to β in A by the path that is the union of these two line segments, as shown below.



The set A is not convex because, for example, $2i$ and -3 cannot be joined by a line segment lying entirely in A.

Solution to Exercise 4.4

Choose any two points α and β in $A \cup B$. If both points belong to the connected set A, then they can be joined by a path lying entirely in A, and this path also lies entirely in $A \cup B$. Similarly, if both points belong to B, then they can be joined by a path lying entirely in $A \cup B$.

Now suppose that one point lies in A and the other lies in B; say, $\alpha \in A$ and $\beta \in B$. Choose any point α_0 in $A \cap B$ (which is possible since $A \cap B \neq \varnothing$). Since A and B are connected, there is a path Γ_α lying entirely in A that joins α to α_0, and there is a path Γ_β lying entirely in B that joins α_0 to β. Therefore the path $\Gamma = \Gamma_\alpha \cup \Gamma_\beta$ joins α to β and it lies entirely in $A \cup B$.

Hence $A \cup B$ is connected.

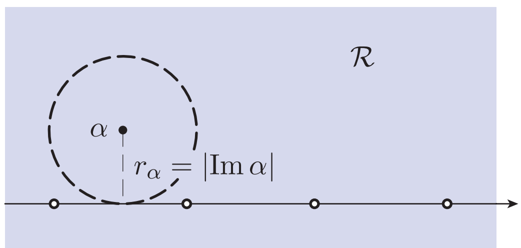
Solution to Exercise 4.5

Clearly, $\mathcal{R} \neq \varnothing$, since $0 \in \mathcal{R}$.

We now prove that $\mathcal{R}$ is open. Let $\alpha \in \mathcal{R}$.

If $\operatorname{Im} \alpha \neq 0$, then take $r_\alpha = |\operatorname{Im} \alpha|$ and note that

$$\{z : |z-\alpha| < r_\alpha\} \subseteq \mathcal{R}.$$



If $\operatorname{Im} \alpha = 0$, then α is real and

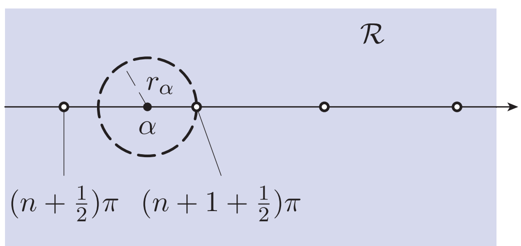
$$\left(n+\frac{1}{2}\right)\pi < \alpha < \left(n+1+\frac{1}{2}\right)\pi,$$

for some integer n. Thus, if

$$r_\alpha = \min\{\alpha - (n+\tfrac{1}{2})\pi, (n+1+\tfrac{1}{2})\pi - \alpha\},$$

then

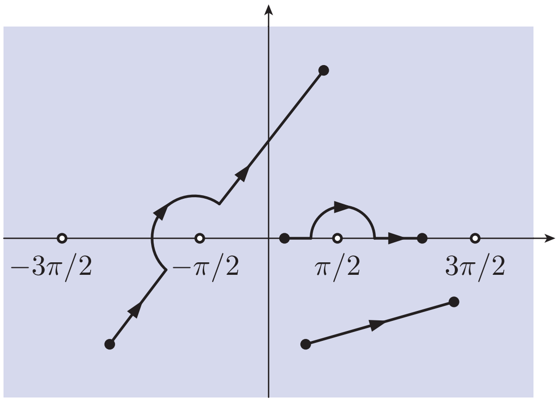
$$\{z : |z-\alpha| < r_\alpha\} \subseteq \mathcal{R}.$$



Hence $\mathcal{R}$ is open.

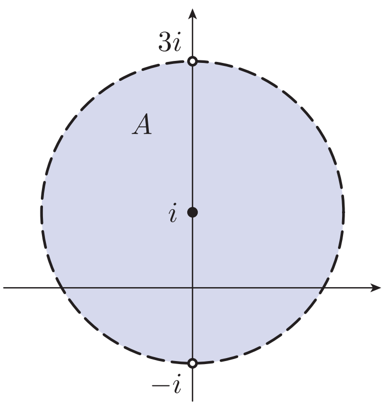
Next we prove that $\mathcal{R}$ is connected. (There are many ways of doing this.)

Any two points of $\mathcal{R}$ can be joined by a line segment, modified if necessary by semicircular arcs to avoid points of $\mathbb{C} - \mathcal{R}$. At most, a finite number of such points will need to be avoided. The figure shows some suitable paths formed in this way.

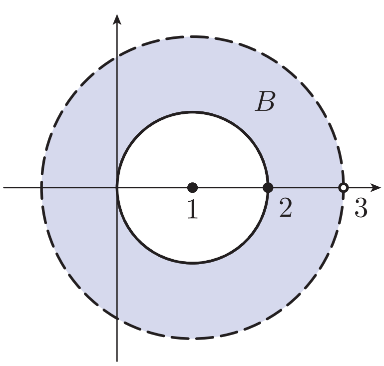


Solution to Exercise 4.6

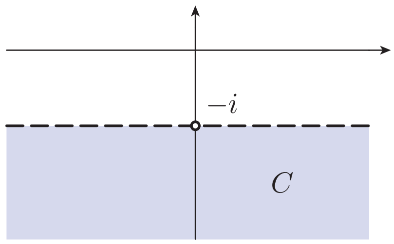
(a) $A = \{z : |z - i| < 2\}$ is open, convex, connected, a region.



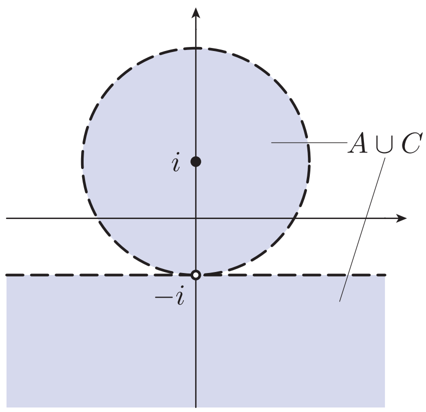
(b) $B = \{z : 1 \leq |z - 1| < 2\}$ is not open, not convex, connected, not a region.



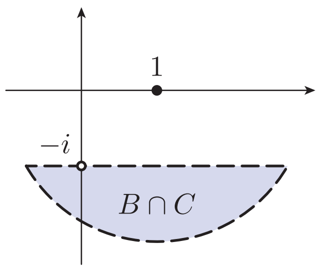
(c) $C = \{z : \operatorname{Im} z < -1\}$ is open, convex, connected, a region.



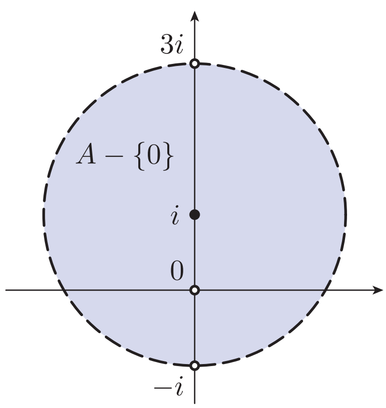
(d) $A \cup C$ is open, not convex, not connected (no path in $A \cup C$ joins 0 and $-2i$, for example), not a region.



(e) $B \cap C$ is open, convex, connected, a region.



(f) $A - \{0\}$ is open, not convex, connected, a region (by Theorem 4.4).



Solution to Exercise 4.7

We prove that $A = \{z : |z - i| < 2\}$ is (a) open, (b) convex, (c) connected, (d) a region.

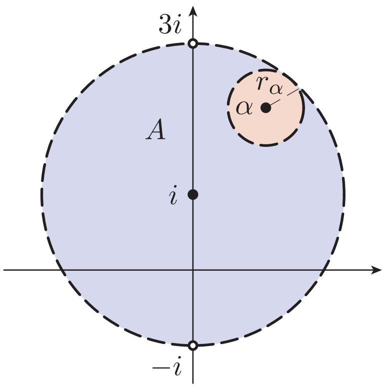
(a) The boundary of A is $\{z : |z - i| = 2\}$. If $\alpha \in A$, then the distance from α to the boundary of A is

$$r_\alpha = 2 - |\alpha - i| > 0.$$

Hence the open disc

$$\{z : |z - \alpha| < r_\alpha\}$$

lies entirely in A (see the figure), so A is open.



(b) Let α and β be any two points of A. Clearly, the line segment from α to β lies entirely within A. Hence A is convex.

(c) Since A is convex, A is connected.

(d) A is non-empty (it contains the point i, for example), open and connected; hence it is a region.

Solution to Exercise 5.1

(a) Since

$$\mathbb{C} - \{z : |z| \geq 1\} = \{z : |z| < 1\}$$

is open, we deduce that $\{z : |z| \geq 1\}$ is closed.

(b) Since

$$\mathbb{C} - \{z : |z| = 1\}$$
$$= \{z : |z| < 1\} \cup \{z : |z| > 1\}$$

is open (by Theorem 4.1(a)), we deduce that $\{z : |z| = 1\}$ is closed.

Solution to Exercise 5.2

(a) Each of the sets

$$E_1 = \{z : \operatorname{Re} z \leq 1\}, \quad E_2 = \{z : \operatorname{Re} z \geq -1\},$$
$$E_3 = \{z : \operatorname{Im} z \leq 1\}, \quad E_4 = \{z : \operatorname{Im} z \geq -1\},$$

is a closed half-plane, and hence

$$\{z : -1 \leq \operatorname{Re} z \leq 1,\ -1 \leq \operatorname{Im} z \leq 1\}$$
$$= E_1 \cap E_2 \cap E_3 \cap E_4$$

is closed, by the corollary to Theorem 5.1.

(b) Each of the sets

$$E_1 = \{z : \operatorname{Im} z \geq 0\}, \quad E_2 = \{z : \operatorname{Im} z \leq 0\}$$

is a closed half-plane, and hence

$$\{z : \operatorname{Im} z = 0\} = E_1 \cap E_2,$$

is closed, by Theorem 5.1(b).

Solution to Exercise 5.3

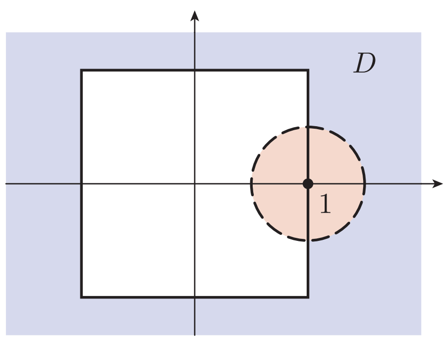
(a) The set $\{z : |z| = 1\}$ is closed, by Exercise 5.1(b); it is bounded because it lies in the closed disc $\{z : |z| \leq 1\}$.

(b) The set $\{z : \operatorname{Im} z = 0\}$ is closed, by Exercise 5.2(b); but it is not bounded.

(c) The set

$$E = \{z : -1 < \operatorname{Re} z < 1,\ -1 < \operatorname{Im} z < 1\}$$

is bounded, since it lies in the closed disc $\{z : |z| \leq \sqrt{2}\}$. However, this set is not closed because its complement $D = \mathbb{C} - E$ is not open. For example, the point 1 lies in D, but no open disc centred at 1 lies entirely in D, as indicated in the figure.



Solution to Exercise 5.4

The set

$$A = \{z : -1 \leq \operatorname{Re} z \leq 1,\ -1 \leq \operatorname{Im} z \leq 1\}$$

is closed, by Exercise 5.2(a). It is bounded because it lies in the closed disc $\{z : |z| \leq \sqrt{2}\}$, for example. Hence A is compact.

Solution to Exercise 5.5

(a) The function $f(z) = e^{1/z}$ is continuous on its domain $\mathbb{C} - \{0\}$, so it is continuous on the circle

$$E = \left\{z : |z| = \tfrac{1}{2}\right\},$$

which is a compact set. Hence f is bounded on E, by the Boundedness Theorem.

Alternatively,

$$\begin{aligned} |f(z)| &= |e^{1/z}| \\ &\le e^{|1/z|} \quad \text{(Exercise 4.2(b) of Unit A2)} \\ &= e^{1/|z|} \\ &= e^2, \quad \text{for } |z| = \tfrac{1}{2}, \end{aligned}$$

so f is bounded on E.

(b) The function $f(z) = \sin z$ is continuous on $\mathbb{C}$, so it is continuous on the closed disc

$$E = \{z : |z| \le 27\},$$

which is a compact set. Hence f is bounded on E, by the Boundedness Theorem.

Alternatively, using the fact that $|e^w| \le e^{|w|}$ (from Exercise 4.2(b) of Unit A2), we have

$$\begin{aligned} |f(z)| &= |\sin z| \\ &= \left|\frac{e^{iz} - e^{-iz}}{2i}\right| \\ &\le \tfrac{1}{2}\left(|e^{iz}| + |e^{-iz}|\right) \quad \text{(Triangle Inequality)} \\ &\le \tfrac{1}{2}\left(e^{|iz|} + e^{|-iz|}\right) \\ &= e^{|z|} \\ &\le e^{27}, \quad \text{for } |z| \le 27, \end{aligned}$$

so f is bounded on E.

(c) The function $f(z) = \left(z^2 + 1\right)/(z - 2i)$ is continuous on $\mathbb{C} - \{2i\}$, so it is continuous on

$$E = \{z : -1 \le \operatorname{Re} z \le 1,\ -1 \le \operatorname{Im} z \le 1\},$$

which is a compact set (by Exercise 5.4). Hence f is bounded on E, by the Boundedness Theorem.

Alternatively, first observe that $|z| \le \sqrt{2}$, for $z \in E$. Also,

$$|f(z)| = \left|\frac{z^2 + 1}{z - 2i}\right|.$$

Applying the Triangle Inequality to the numerator, we obtain, for $z \in E$,

$$|z^2 + 1| \le |z|^2 + 1 \le (\sqrt{2})^2 + 1 = 3.$$

Applying the backwards form of the Triangle Inequality to the denominator, we obtain, for $z \in E$,

$$|z - 2i| \ge |2i| - |z| \ge 2 - \sqrt{2}.$$

Combining the inequalities for numerator and denominator gives $|f(z)| \le 3/(2 - \sqrt{2})$, for $z \in E$, so f is bounded on E.

Solution to Exercise 5.6

(a) $\operatorname{int} A = \{z : |z| < 1\}$
$\operatorname{ext} A = \{z : |z| > 1\}$
$\partial A = \{z : |z| = 1\}$

(b) $\operatorname{int} A = \{x + iy : x < 0\}$
$\operatorname{ext} A = \{x + iy : x > 0\}$
$\partial A = \{x + iy : x = 0\}$

(c) $\operatorname{int} A = \mathbb{C} - \{0\}$
$\operatorname{ext} A = \varnothing$
$\partial A = \{0\}$

(d) $\operatorname{int} A = \varnothing$
$\operatorname{ext} A = \mathbb{C} - \{0\}$
$\partial A = \{0\}$

Solution to Exercise 5.7

The sets A, B, $B \cap C$ and $A - \{0\}$ are bounded, for each is contained in the closed disc $\{z : |z| \le 3\}$. The sets C and $A \cup C$ are not bounded.

Solution to Exercise 5.8

(a) The complement of $E = \{z : |z - i| \ge 2\}$ is $\{z : |z - i| < 2\}$, which is open (by Exercise 4.7). Hence E is a closed set.

(b) The complement of

$$E = \{z : |z - 1| < 1 \text{ or } |z - 1| \ge 2\}$$

is

$$D = \{z : 1 \le |z - 1| < 2\},$$

which is not open, because any disc centred at the point $0 \in D$ (for example) contains points that are not in D. Hence E is not a closed set.

(c) The complement of $E = \{z : \operatorname{Im} z \le -1\}$ is $\{z : \operatorname{Im} z > -1\}$, which is open. Hence E is a closed set.

Solution to Exercise 5.9

(a) The function $f(z) = \sinh z$ is continuous on its domain $\mathbb{C}$, so it is continuous on $E = \{z : |z| \leq 1\}$, which is a compact set. So f is bounded on E, by the Boundedness Theorem.

(b) The Boundedness Theorem is inapplicable because the function $f(z) = \operatorname{Log} z$ is not defined at 0, which is in $E = \{z : |z| \leq 1\}$, so it is not continuous on E.

(c) The Boundedness Theorem is inapplicable because $E = \{z : \operatorname{Re} z \geq 1\}$ is not a compact set (it is not bounded).

(d) The function $f(z) = 1/z$ is continuous on its domain $\mathbb{C} - \{0\}$, so it is continuous on $E = \{z : 1 \leq |z| \leq 2\}$, which is a compact set. So f is bounded on E, by the Boundedness Theorem.

(e) The Boundedness Theorem is inapplicable because $E = \{z : 0 < |z| \leq 2\}$ is not a compact set (it is not closed).

(f) The Boundedness Theorem is inapplicable because $E = \{z : |z| \geq 1\}$ is not a compact set (it is not bounded). However, note that f is actually bounded (by 1) on E.

Solution to Exercise 5.10

$\operatorname{int} A = \{z : |z - 1| < 1 \text{ or } |z - 1| > 2\}$
$\operatorname{ext} A = \{z : 1 < |z - 1| < 2\}$
$\partial A = \{z : |z - 1| = 1 \text{ or } |z - 1| = 2\}$

Unit A4
Differentiation

Introduction

The derivative of a real function f at a point c is the gradient of the tangent to the graph of f at c. This gradient is calculated by finding the gradient of the chord joining the point $(c, f(c))$ to a (nearby) point $(x, f(x))$, and taking the limit as x approaches c (Figure 0.1).

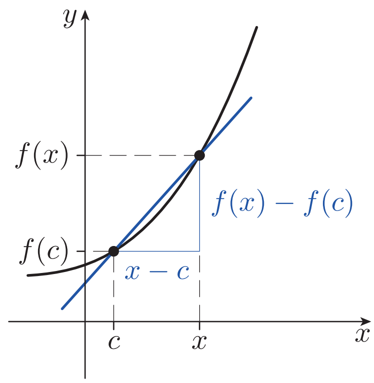


Figure 0.1 A chord between two points on a graph

Now, the gradient of the chord is equal to the ratio

$$\frac{f(x) - f(c)}{x - c}.$$

This ratio is often called the *difference quotient* for f at c, and its limit as x tends to c provides a formal definition of the (real) derivative of f at c, denoted by $f'(c)$. Thus

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

In the case of complex functions, it is difficult to think about derivatives in terms of gradients of tangents, since the graph of a complex function is not drawn in two dimensions. Instead we define the derivative of a complex function directly in terms of difference quotients, using the notion of complex limits discussed in the previous unit.

Fortunately, the derivatives of many complex functions turn out to have the same form as those of the corresponding real functions. For example, the derivative of the complex sine function is the complex cosine function, and the complex exponential function is its own derivative. On the other hand, the complex modulus function fails to be differentiable at any point of $\mathbb{C}$, even though the real modulus function (Figure 0.2) is differentiable at every point of $\mathbb{R} - \{0\}$. This reflects the fact that complex differentiation imposes a much stronger condition on functions than does real differentiation. Indeed, as the module progresses, you will see that differentiable complex functions have remarkably pleasant properties. For example, if a complex function can be differentiated once throughout a region, then it can be differentiated any number of times. There is no equivalent result for real functions.

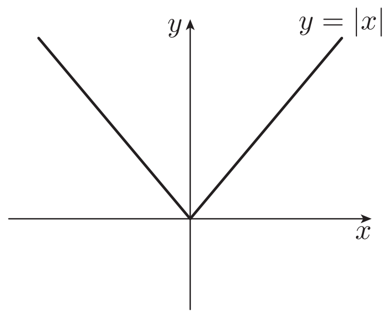


Figure 0.2 Graph of $y = |x|$

In Section 1 we define *complex differentiation* and show how the definition can be used to establish whether a function is differentiable. By introducing rules for combining differentiable functions, we show how complex polynomial and rational functions can be differentiated just as in the real case. At the end of Section 1, we give a geometric interpretation of complex differentiation by introducing the idea of a complex scale factor.

In Section 2 we introduce the concept of *partial differentiation* for real functions of two real variables, and use it to establish a relationship between complex differentiation and real differentiation. This relationship sometimes enables us to differentiate a complex function using real derivatives. Indeed, at the end of the section we use this approach to show that the complex exponential function is its own derivative.

In Section 3 we introduce rules for differentiating composite and inverse functions, and show how they can be used to differentiate the principal logarithm and power functions. We also introduce a Restriction Rule that enables us to cut away unwanted parts from the domain of a function without affecting the differentiability of the function at the remaining points.

Finally, in Section 4 we concentrate on the derivatives of parametrisations of paths. We introduce the notion of a *smooth* path, and use it to pursue the geometric interpretation of derivative begun in Section 1. In particular, we show that a differentiable complex function with non-zero derivative preserves angles between paths.

Unit guide

Sections 1 and 3 contain the complex versions of many results that will be familiar to you from your study of real differentiation. This should help you to remember the results. There are, however, important differences between real and complex differentiation, and you should study the sections with this in mind.

The only information that you will need to know from Section 2 in order to study Section 3 is that the complex exponential function is its own derivative. However, the material from Section 2 will be essential in later units.

The work on paths in Section 4 is important, and will be used throughout the module. In particular, it will be used in Unit B1 to define complex integration.

1 Derivatives of complex functions

After working through this section, you should be able to:

- use the definition of *derivative* to show that a given function is differentiable, and to find its derivative
- use the Combination Rules for differentiation to differentiate polynomial and rational functions
- use various strategies to show that a given function is not differentiable at a point
- interpret the derivative of a complex function at a point as a rotation and a scaling of a small disc centred at the point.

1.1 Defining differentiable functions

As with limits and continuity, the way in which the derivative of a complex function is defined is similar to the real case. Thus a complex function is said to have a *derivative* at a point $\alpha \in \mathbb{C}$ if the **difference quotient**, defined by

$$\frac{f(z) - f(\alpha)}{z - \alpha},$$

tends to a limit as z tends to α. Equivalently, it is sometimes more convenient to replace z by $\alpha + h$, and examine the corresponding limit as h tends to 0. The difference quotient then has the form

$$\frac{f(\alpha + h) - f(\alpha)}{h},$$

where h is a complex number. The equivalence of these two limits can be justified by noting that if $z = \alpha + h$, then '$z \to \alpha$' is equivalent to '$h \to 0$'.

Definitions

Let f be a complex function whose domain contains the point α. Then the **derivative of $\boldsymbol{f}$ at $\boldsymbol{\alpha}$** is

$$\lim_{z \to \alpha} \frac{f(z) - f(\alpha)}{z - \alpha} \quad \left(\text{or } \lim_{h \to 0} \frac{f(\alpha + h) - f(\alpha)}{h}\right),$$

provided that this limit exists. If it does exist, then f is **differentiable at $\boldsymbol{\alpha}$**. If f is differentiable at *every* point of a set A, then f is **differentiable on $\boldsymbol{A}$**. A function is **differentiable** if it is differentiable on its domain.

The derivative of f at α is denoted by $f'(\alpha)$, and the function

$$f' : z \longmapsto f'(z)$$

is called the **derivative of $\boldsymbol{f}$**. The domain of f' is the set of all complex numbers at which f is differentiable.

The function f' is sometimes called the *derived function* of f.

Remarks

1. The existence of the limit

$$\lim_{z \to \alpha} \frac{f(z) - f(\alpha)}{z - \alpha}$$

implicitly requires the domain of f to contain α as one of its limit points. This always holds if the domain of f is a region. (For the definition of limit point, see Subsection 3.1 of Unit A3.)

2. The derivative $f'(z)$ is sometimes written as $\dfrac{df}{dz}(z)$ or $\dfrac{d}{dz}(f(z))$.

3. Some other texts use the phrase *complex derivative* in place of *derivative* to draw a distinction with the standard real derivative of a function $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ (which we will not need).

In certain cases it is easy to find the derivative of a function directly from the definition above.

Example 1.1

Use the definition of derivative to find the derivative of the function $f(z) = z^2$.

Solution

The domain of $f(z) = z^2$ is the whole of $\mathbb{C}$, so let α be an arbitrary point of $\mathbb{C}$. Then

$$\begin{aligned} f'(\alpha) &= \lim_{z\to\alpha} \frac{f(z) - f(\alpha)}{z - \alpha} \\ &= \lim_{z\to\alpha} \frac{z^2 - \alpha^2}{z - \alpha} \\ &= \lim_{z\to\alpha} (z + \alpha). \end{aligned}$$

Now $z \longmapsto z + \alpha$ is a basic continuous function, continuous at α, so we see (from Theorem 3.1 of Unit A3) that $f'(\alpha) = \alpha + \alpha = 2\alpha$.

Since α is an arbitrary complex number, the derivative of f is the function $f'(z) = 2z$. Its domain is the whole of $\mathbb{C}$.

Notice the way in which the troublesome $z - \alpha$ term cancels from the numerator and the denominator in the calculation of $f'(\alpha)$ in the preceding example. This often happens when you calculate derivatives directly from the definition.

Exercise 1.1

Use the definition of derivative to find the derivative of

(a) the constant function $f(z) = 1$ (b) the function $f(z) = z$.

Example 1.1 and Exercise 1.1 show that the functions $f(z) = 1$, $f(z) = z$ and $f(z) = z^2$ are differentiable on the whole of $\mathbb{C}$. Functions that have this property are given a special name.

Definition

A function is **entire** if it is differentiable on the whole of $\mathbb{C}$.

Not all functions are entire; indeed, many interesting aspects of complex analysis arise from functions that fail to be differentiable at various points of $\mathbb{C}$.

Exercise 1.2

Use the definition of derivative to find the derivative of the function $f(z) = 1/z$. Explain why f is not entire.

Although the function $f(z) = 1/z$ is not entire, it is differentiable on the whole of its domain $\mathbb{C} - \{0\}$. This domain is a region because it is obtained by removing the point 0 from $\mathbb{C}$. (The removal of a point from a region leaves a region, by Theorem 4.4 of Unit A3.) As the module progresses, you will discover that regions provide an excellent setting for analysing the properties of differentiable functions. We therefore make the following definitions.

Definitions

A function that is differentiable on a region $\mathcal{R}$ is said to be **analytic on $\mathcal{R}$**. If the domain of a function f is a region, and if f is differentiable on its domain, then f is said to be **analytic**. A function is **analytic at a point α** if it is differentiable on a region containing α.

It follows immediately from the definition that if a function is analytic on a region $\mathcal{R}$, then it is automatically analytic at each point of $\mathcal{R}$.

Notice that a function can have a derivative at a point without being analytic at the point. For example, in the next section we will ask you to show that the function $g(z) = |z|^2$ has a derivative at 0, but at no other point. This means that there is no region on which g is differentiable, and hence no point at which g is analytic.

By contrast, $f(z) = 1/z$ is analytic at every point of its domain. It is an analytic function, and it is analytic on *any* region that does not contain 0. Three such regions are illustrated in Figure 1.1.

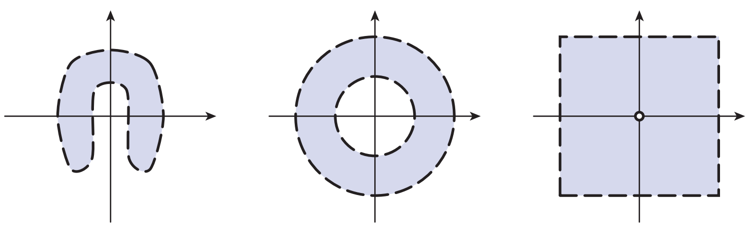


Figure 1.1 Three regions on which $f(z) = 1/z$ is analytic

An appropriate choice of region can often simplify the analysis of complex functions.

Exercise 1.3

Classify each of the following statements as True or False.

(a) An entire function is analytic at every point of $\mathbb{C}$.

(b) If a function is differentiable at each point of a set, then it is analytic on that set.

There is a close connection between differentiation and continuity. The function $f(z) = 1/z$, for example, is not only differentiable, but also continuous on its domain. This is no accident for, as in real analysis, *differentiability implies continuity*.

Theorem 1.1

Let f be a complex function that is differentiable at α. Then f is continuous at α.

Proof Let f be differentiable at α; then

$$\lim_{z\to\alpha} \frac{f(z) - f(\alpha)}{z - \alpha} = f'(\alpha).$$

To prove that f is continuous at α, we will show that $f(z) \to f(\alpha)$ as $z \to \alpha$. We do this by proving the equivalent result that $f(z) - f(\alpha) \to 0$ as $z \to \alpha$.

By the Product Rule for limits of functions, we have

$$\begin{aligned}\lim_{z\to\alpha} (f(z) - f(\alpha)) &= \lim_{z\to\alpha} \left(\frac{f(z) - f(\alpha)}{z - \alpha} \right) \times \lim_{z\to\alpha} (z - \alpha) \\ &= f'(\alpha) \times 0 = 0.\end{aligned}$$

Hence $f(z) \to f(\alpha)$ as $z \to \alpha$, so f is continuous at α. ■

In fact, differentiability implies more than continuity. Continuity asserts that for all z close to α, $f(z)$ is close to $f(\alpha)$. For *differentiable* functions, this 'closeness' has the 'linear' form described in the following theorem.

Theorem 1.2 Linear Approximation Theorem

Let f be a complex function that is differentiable at α. Then f can be approximated near α by a linear polynomial. More precisely,

$$f(z) = f(\alpha) + (z - \alpha)f'(\alpha) + e(z),$$

where e is an 'error function' satisfying $e(z)/(z - \alpha) \to 0$ as $z \to \alpha$.

Informally speaking, the statement '$e(z)/(z - \alpha) \to 0$ as $z \to \alpha$' means that '$e(z)$ tends to zero faster than $z - \alpha$ does'.

Proof We have to show that the function e defined by

$$e(z) = f(z) - f(\alpha) - (z-\alpha)f'(\alpha)$$

satisfies $e(z)/(z-\alpha) \to 0$ as $z \to \alpha$.

Dividing $e(z)$ by $z-\alpha$ and letting z tend to α, we obtain

$$\begin{aligned}\lim_{z\to\alpha} \frac{e(z)}{z-\alpha} &= \lim_{z\to\alpha}\left(\frac{f(z)-f(\alpha)}{z-\alpha} - f'(\alpha)\right)\\ &= f'(\alpha) - f'(\alpha) = 0,\end{aligned}$$

as required. ■

Theorems 1.1 and 1.2 are often used to investigate the properties of differentiable functions. An illustration of this occurs in the next subsection, where Theorem 1.1 is used in a proof of the Combination Rules for differentiation. Later in this section we use Theorem 1.2 to give a geometric interpretation of complex differentiation.

1.2 Combining differentiable functions

It would be tedious if we had to use the definition of the derivative every time we needed to differentiate a function. Fortunately, once the derivatives of simple functions like $z \longmapsto 1$ and $z \longmapsto z$ are known, we can find the derivatives of other more complicated functions by applying the following theorem.

Theorem 1.3 Combination Rules for Differentiation

Let f and g be complex functions with domains A and B, respectively, and let α be a limit point of $A \cap B$. If f and g are differentiable at α, then

(a) **Sum Rule** $f+g$ is differentiable at α, and

$$(f+g)'(\alpha) = f'(\alpha) + g'(\alpha)$$

(b) **Multiple Rule** λf is differentiable at α, for $\lambda \in \mathbb{C}$, and

$$(\lambda f)'(\alpha) = \lambda f'(\alpha)$$

(c) **Product Rule** fg is differentiable at α, and

$$(fg)'(\alpha) = f'(\alpha)g(\alpha) + f(\alpha)g'(\alpha)$$

(d) **Quotient Rule** f/g is differentiable at α (provided that $g(\alpha) \neq 0$), and

$$\left(\frac{f}{g}\right)'(\alpha) = \frac{g(\alpha)f'(\alpha) - f(\alpha)g'(\alpha)}{(g(\alpha))^2}.$$

We remark that if the domains A and B in Theorem 1.3 are regions, then every point of $A \cap B$ is a limit point of A and of B.

In addition to these rules, there is a corollary to Theorem 1.3, known as the Reciprocal Rule, which is a special case of the Quotient Rule.

Corollary Reciprocal Rule for Differentiation

Let f be a function that is differentiable at α. If $f(\alpha) \neq 0$, then $1/f$ is differentiable at α, and

$$\left(\frac{1}{f}\right)'(\alpha) = -\frac{f'(\alpha)}{(f(\alpha))^2}.$$

The proof of the Combination Rules for differentiation uses the Combination Rules for limits of functions, discussed in Unit A3. In the next example we illustrate the method by proving the Product Rule for differentiation. We use the Sum, Product and Multiple Rules for limits of functions, and we also use the fact that if a function g is differentiable at α, then it is continuous at α, so $\lim\limits_{z\to\alpha} g(z) = g(\alpha)$.

Example 1.2

Prove the Product Rule for differentiation.

Solution

Let $F = fg$. Then

$$\begin{aligned}
&\lim_{z\to\alpha} \frac{F(z) - F(\alpha)}{z-\alpha} \\
&= \lim_{z\to\alpha} \frac{f(z)g(z) - f(\alpha)g(\alpha)}{z-\alpha} \\
&= \lim_{z\to\alpha} \frac{(f(z) - f(\alpha))g(z) + f(\alpha)(g(z) - g(\alpha))}{z-\alpha} \\
&= \left(\lim_{z\to\alpha} \frac{f(z)-f(\alpha)}{z-\alpha}\right)\left(\lim_{z\to\alpha} g(z)\right) + f(\alpha)\left(\lim_{z\to\alpha} \frac{g(z)-g(\alpha)}{z-\alpha}\right) \\
&= f'(\alpha)g(\alpha) + f(\alpha)g'(\alpha).
\end{aligned}$$

The proofs of the other Combination Rules are similar. We ask you to prove the Sum and Multiple Rules in Exercise 1.4, and the Quotient Rule in Exercise 1.12.

Exercise 1.4

Prove the following rules for differentiation.

(a) Sum Rule (b) Multiple Rule

The Combination Rules enable us to differentiate any polynomial or rational function. (Recall that a rational function is the quotient of two polynomial functions.)

For example, since the function $f(z) = z$ is entire with derivative $f'(z) = 1$, we can use the Product Rule repeatedly to show that the function

$$f(z) = z^n \quad (z \in \mathbb{C})$$

is entire, and that its derivative is

$$f'(z) = nz^{n-1} \quad (z \in \mathbb{C}).$$

(This result can be proved formally using the Principle of Mathematical Induction.) Next, we can use this fact, together with the Sum and Multiple Rules, to prove that any polynomial function is entire, and that its derivative is obtained by differentiating the polynomial function term by term. For example,

$$\text{if } f(z) = z^4 - 3z^2 + 2z + 1, \text{ then } f'(z) = 4z^3 - 6z + 2.$$

In general, we have the following corollary to Theorem 1.3.

Corollary Differentiating Polynomial Functions

Let p be the polynomial function

$$p(z) = a_n z^n + \cdots + a_2 z^2 + a_1 z + a_0 \quad (z \in \mathbb{C}),$$

where $a_0, a_1, \ldots, a_n \in \mathbb{C}$ and $a_n \neq 0$. Then p is entire with derivative

$$p'(z) = n a_n z^{n-1} + \cdots + 2a_2 z + a_1 \quad (z \in \mathbb{C}).$$

Since a rational function is a quotient of two polynomial functions, it follows from the corollary on differentiating polynomial functions and the Quotient Rule that a rational function is differentiable at all points where its denominator is non-zero; that is, at all points of its domain.

Example 1.3

Find the derivative of

$$f(z) = \frac{2z^2 + z}{z^2 + 1},$$

and specify its domain.

Solution

By the corollary on differentiating polynomial functions, the derivative of $z \longmapsto 2z^2 + z$ is

$$z \longmapsto 4z + 1,$$

and the derivative of $z \longmapsto z^2 + 1$ is

$$z \longmapsto 2z.$$

Provided that $z^2 + 1$ is non-zero, we can apply the Quotient Rule to obtain

$$f'(z) = \frac{(z^2+1)(4z+1) - (2z^2+z)(2z)}{(z^2+1)^2} = \frac{-z^2+4z+1}{(z^2+1)^2}.$$

Since $z^2 + 1$ is non-zero everywhere apart from i and $-i$, it follows that the domain of f' is $\mathbb{C} - \{i, -i\}$.

Exercise 1.5

Find the derivative of each of the following functions. In each case specify the domain of the derivative.

(a) $f(z) = z^4 + 3z^3 - z^2 + 4z + 2$ (b) $f(z) = \dfrac{z^2 - 4z + 2}{z^2 + z + 1}$

So, any rational function is differentiable on the whole of its domain. What is more, this domain must be a region because it is obtained by removing a finite number of points (zeros of the denominator) from $\mathbb{C}$.

Corollary

Any rational function is analytic.

A particularly simple example of a rational function is $f(z) = 1/z^n$, where n is a positive integer. This can be differentiated by means of the Reciprocal Rule:

$$f'(z) = -\frac{nz^{n-1}}{(z^n)^2} = -nz^{-n-1}.$$

If k is used to denote the negative integer $-n$, then we can write $f(z) = z^k$ and $f'(z) = kz^{k-1}$. In this form, it is apparent that the formula for differentiating a negative integer power is the same as the formula for differentiating a positive integer power. The only difference is that for negative powers, 0 is excluded from the domain. We state these observations as a final corollary to Theorem 1.3.

Corollary

Let $k \in \mathbb{Z} - \{0\}$. The function $f(z) = z^k$ has derivative

$$f'(z) = kz^{k-1}.$$

The domain of f' is $\mathbb{C}$ if $k > 0$ and $\mathbb{C} - \{0\}$ if $k < 0$.

1.3 Non-differentiability

In Theorem 1.1 you saw that *differentiability implies continuity*. An immediate consequence of this is the following test for non-differentiability.

Strategy A for non-differentiability

If f is discontinuous at α, then f is not differentiable at α.

Example 1.4

Show that there are no points of the negative real axis at which the function $f(z) = \sqrt{z}$ is differentiable.

Solution

In Exercise 2.8 of Unit A3 you saw that the function $f(z) = \sqrt{z}$ is discontinuous at all points of the negative real axis. It follows that there are no points of the negative real axis at which f is differentiable.

Exercise 1.6

Show that there are no points of the negative real axis at which the principal logarithm function

$$\operatorname{Log} z = \log |z| + i \operatorname{Arg} z$$

is differentiable.

The converse of Theorem 1.1 is not true; if a function is continuous at a point, then it does not follow that it is differentiable at the point. A particularly striking illustration of this is provided by the modulus function $f(z) = |z|$. This is continuous on the whole of $\mathbb{C}$ (as you saw in Exercise 2.2(f) of Unit A3) and yet, as you will see, it fails to be differentiable at any point of $\mathbb{C}$.

Since $f(z) = |z|$ is continuous, Strategy A cannot be used to show that f is not differentiable at a given point α. Instead we return to the definition of derivative and show that the difference quotient for f fails to have a limit.

In general, if the domain A of a function f contains α as one of its limit points, then the existence of the limit

$$\lim_{z\to\alpha} \frac{f(z) - f(\alpha)}{z - \alpha}$$

means that for each sequence (z_n) in $A - \{\alpha\}$ that converges to α,

$$\lim_{n\to\infty} \frac{f(z_n) - f(\alpha)}{z_n - \alpha}$$

exists, and has a value that is independent of the sequence (z_n).

So, if two such sequences (z_n) and (z'_n) can be found for which

$$\lim_{n\to\infty} \frac{f(z_n) - f(\alpha)}{z_n - \alpha} \neq \lim_{n\to\infty} \frac{f(z'_n) - f(\alpha)}{z'_n - \alpha},$$

then f cannot be differentiable at α. (See the strategy for proving that a limit does not exist, in Subsection 3.1 of Unit A3.)

In the next example, you will see that $f(z) = |z|$ is not differentiable at 0. This result should not surprise you because the real modulus function is not differentiable at 0. Indeed, the proof is identical to that of the real case.

Example 1.5

Prove that $f(z) = |z|$ is not differentiable at 0.

Solution

We need to find two sequences (z_n) and (z'_n) that converge to 0 which, when substituted into the difference quotient, yield sequences with different limits. A simple choice is to pick sequences (z_n) and (z'_n) that approach 0 along the real axis: one from the right, and one from the left, as shown in Figure 1.2.

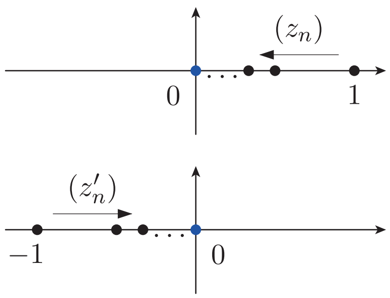


Figure 1.2 Sequences converging to 0 from the right and left

There is no point in picking sequences that are more complicated than they need to be, so let $z_n = 1/n$, $n = 1, 2, \ldots$. Then

$$\lim_{n\to\infty} \frac{|z_n| - |0|}{z_n - 0} = \lim_{n\to\infty} \frac{1/n}{1/n} = 1.$$

Now let $z'_n = -1/n$, $n = 1, 2, \ldots$. Then

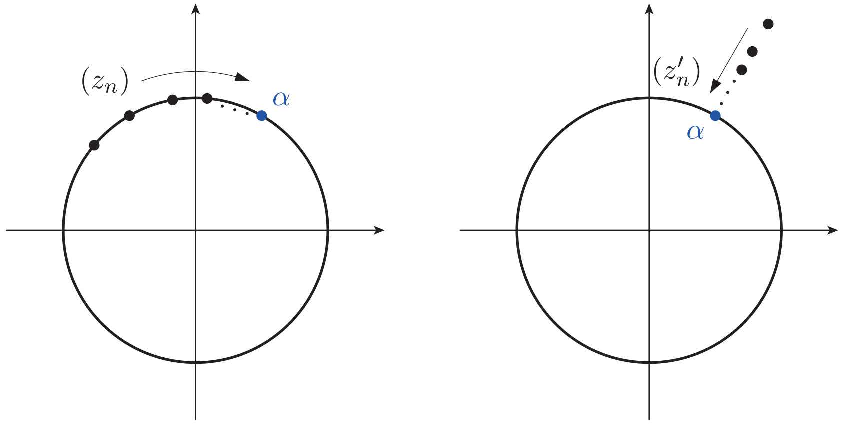
$$\lim_{n\to\infty} \frac{|z'_n| - |0|}{z'_n - 0} = \lim_{n\to\infty} \frac{1/n}{-1/n} = -1.$$

Since the two limits do not agree, the difference quotient does not have a limit as z tends to 0. It follows that $f(z) = |z|$ is *not* differentiable at 0.

The next exercise asks you to extend the method used in Example 1.5 to show that $f(z) = |z|$ is not differentiable at *any* point of $\mathbb{C}$.

Exercise 1.7

Let α be any non-zero complex number, and consider the circle through α centred at the origin. By choosing one sequence (z_n) that approaches α along the circumference of the circle, and another sequence (z'_n) that approaches α along the ray from 0 through α, prove that $f(z) = |z|$ is not differentiable at α.



The modulus function illustrates an important difference between real and complex differentiation. When the modulus function is treated as a *real* function, the limit of its difference quotient has to be taken along the real line. But when treated as a *complex* function, the limit of the difference quotient is required to exist however the limit is taken. This explains why the real modulus function is differentiable at all non-zero real points, whereas the complex modulus function fails to be differentiable at any point of $\mathbb{C}$. More generally, it shows that complex differentiability is a much stronger condition than real differentiability.

In Exercise 1.7 you were asked to prove that the modulus function fails to be differentiable by observing that its behaviour along the circumference of a circle centred at 0 is different from its behaviour along a ray. Similar observations can be applied to other functions. For example, in the next exercise you may find it helpful to notice that directions of paths parallel to the imaginary axis are reversed by the function $f(z) = \overline{z}$, whereas directions of paths parallel to the real axis are left unchanged (Figure 1.3).

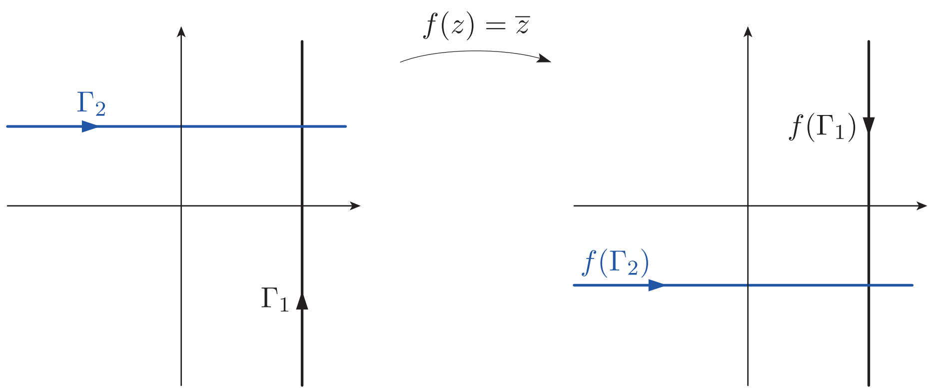


Figure 1.3 Images of horizontal and vertical lines under $f(z) = \overline{z}$

Exercise 1.8

Show that there are no points of $\mathbb{C}$ at which the complex conjugate function $f(z) = \overline{z}$ is differentiable.

For some functions f, you may be able to find a sequence (z_n) that converges to α for which the sequence

$$w_n = \frac{f(z_n) - f(\alpha)}{z_n - \alpha}, \quad n = 1, 2, \ldots, \tag{1.1}$$

is divergent (using the strategy for proving that a limit does not exist, from Subsection 3.1 of Unit A3). In such cases, there is no need to look for a second sequence.

Example 1.6

Show that the function $f(z) = \sqrt{z}$ is not differentiable at 0.

Solution

Strategy A cannot be used here, since f is continuous at 0. Instead we look for a sequence (z_n) that converges to 0 for which the sequence (1.1) is divergent. To make the square roots easy to handle, let $z_n = 1/n^2$, $n = 1, 2, \ldots$. Then

$$\frac{f(z_n) - f(0)}{z_n - 0} = \frac{\sqrt{1/n^2} - \sqrt{0}}{1/n^2 - 0} = \frac{1/n}{1/n^2} = n.$$

This sequence tends to infinity, and is therefore divergent. It follows that f is not differentiable at 0.

The methods exemplified above for showing that a function is not differentiable at a given point can be summarised as follows.

Strategy B for non-differentiability

To prove that a function f is not differentiable at α, apply the strategy for proving that a limit does not exist to the difference quotient

$$\frac{f(z) - f(\alpha)}{z - \alpha}.$$

If you think that a given function is *not* differentiable, then you should try to apply Strategy A or Strategy B. A third strategy for proving that f is not differentiable at a point appears in Subsection 2.1. If, on the other hand, you think that the function *is* differentiable, then you should try to find the derivative.

Exercise 1.9

Decide whether each of the following functions is differentiable at i. If it is, then find its derivative at i.

(a) $f(z) = \operatorname{Re} z$ (b) $f(z) = 2z^2 + 3z + 5$ (c) $f(z) = \begin{cases} z, & \operatorname{Re} z < 0 \\ 4, & \operatorname{Re} z \geq 0 \end{cases}$

1.4 Higher-order derivatives

In Exercise 1.2 you saw that the function $f(z) = 1/z$ has derivative $f'(z) = -1/z^2$, a result that you can also obtain using the Reciprocal Rule. If you now apply the Reciprocal Rule to the derivative $f'(z) = -1/z^2$, then you obtain a function

$$(f')'(z) = \frac{2}{z^3} \quad (z \neq 0).$$

In general, for a differentiable function f, the function $(f')'$ is called the **second derivative of f**, and is denoted by f''. Continued differentiation gives the so-called **higher-order derivatives of f**. These are denoted by $f'', f''', f'''', \ldots$, and the values $f''(\alpha), f'''(\alpha), f''''(\alpha), \ldots$, are called the **higher-order derivatives of f at α**.

Since the dashes in this notation can be rather cumbersome, we often indicate the order of the derivative by a number in brackets. Thus $f^{(2)}, f^{(3)}, f^{(4)}, \ldots$ mean the same as $f'', f''', f'''', \ldots$, respectively. Here the brackets in $f^{(4)}$ are needed to avoid confusion with the fourth power of f.

When we wish to discuss a derivative of general order, we will refer to the **nth derivative $f^{(n)}$ of f**. It is often possible to find a formula for the nth derivative in terms of n. For example, if $f(z) = 1/z$, then

$$f''(z) = \frac{2}{z^3}, \quad f'''(z) = \frac{-2 \times 3}{z^4}, \quad f^{(4)}(z) = \frac{2 \times 3 \times 4}{z^5}, \quad \ldots,$$

so the nth derivative is given by

$$f^{(n)}(z) = \frac{(-1)^n n!}{z^{n+1}}.$$

(This can be proved formally by the Principle of Mathematical Induction.)

One interesting feature about this formula is that the domain $\mathcal{R} = \mathbb{C} - \{0\}$ remains the same, no matter how often the function f is differentiated. This is a special case of a much more general result which states that: *a function that is analytic on a region $\mathcal{R}$ has derivatives of all orders on $\mathcal{R}$*. We will establish this remarkable fact and explore it in more detail in Book B, but for the rest of this unit we confine our attention to first derivatives. We continue to do this in the next subsection by giving a geometric interpretation of the first derivative.

1.5 A geometric interpretation of derivatives

As we mentioned in the Introduction, the derivative of a *real* function is often pictured geometrically as the gradient of the graph of the function. This interpretation is useful in real analysis, but it is of little use in complex analysis, since the graph of a complex function is not two-dimensional.

Fortunately, there is another way of interpreting derivatives that works for complex functions. If a complex function f is differentiable at a point α, then any point z close to α is mapped by f to a point $f(z)$ close to $f(\alpha)$.

Indeed, by the Linear Approximation Theorem,

$$f(z) = f(\alpha) + (z-\alpha)f'(\alpha) + e(z),$$

where $e(z)/(z-\alpha) \to 0$ as $z \to \alpha$. So if $f'(\alpha) \neq 0$, then, *to a close approximation*,

$$f(z) - f(\alpha) \approx f'(\alpha)(z-\alpha).$$

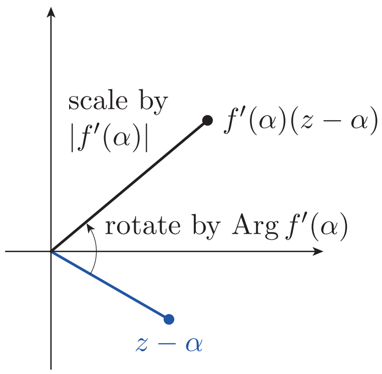


Figure 1.4 Scaling and rotating $z-\alpha$

Multiplication of $z-\alpha$ by $f'(\alpha)$ has the effect of scaling $z-\alpha$ by the factor $|f'(\alpha)|$ and rotating it about 0 through the angle Arg $f'(\alpha)$; see Figure 1.4. We refer to $f'(\alpha)$ as a *complex scale factor*, because it causes both a scaling and a rotation.

We can rewrite the equation above as

$$f(z) \approx f(\alpha) + f'(\alpha)(z-\alpha). \tag{1.2}$$

From this we see that $f(z)$ is obtained by scaling and rotating the vector $z-\alpha$ based at $f(\alpha)$ by the complex scale factor $f'(\alpha)$, as illustrated in Figure 1.5.

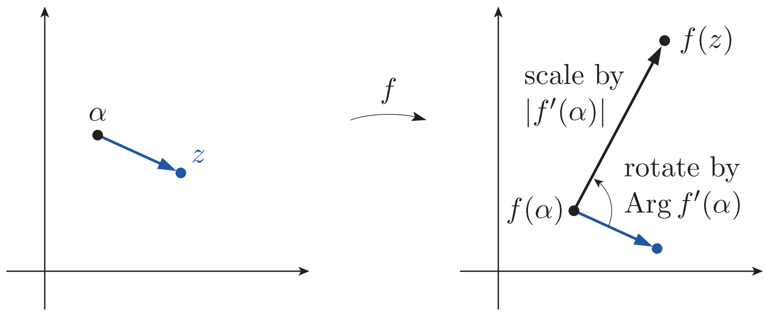


Figure 1.5 Interpreting a derivative as a complex scale factor

Another useful way to picture how f behaves geometrically is to consider the effect it has on a small disc centred at α (still assuming that $f'(\alpha) \neq 0$). From equation (1.2), we see that, to a close approximation, a small disc centred at α is mapped to a small disc centred at $f(\alpha)$. In the process, the disc is rotated through the angle Arg $f'(\alpha)$, and it is scaled by the factor $|f'(\alpha)|$ (see Figure 1.6). As usual, the rotation is anticlockwise if Arg $f'(\alpha)$ is positive, and clockwise if it is negative.

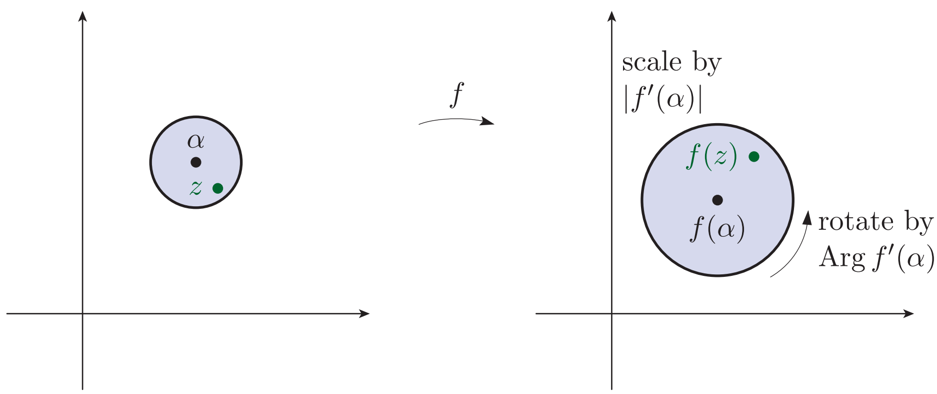


Figure 1.6 The approximate image of a disc centred at a point α, where $f'(\alpha) \neq 0$

The geometric interpretation of derivatives is more complicated if $f'(\alpha) = 0$, and we do not discuss it here (see Unit C2).

Example 1.7

Using the notion of a complex scale factor, describe what happens to points close to $1 + i$ under the function $f(z) = 1/z$.

Solution

To a close approximation, a small disc centred at $1 + i$ is mapped by f to a small disc centred at

$$f(1+i) = 1/(1+i) = \tfrac{1}{2}(1-i).$$

In the process, the disc is scaled by the factor $|f'(1+i)|$ and rotated through the angle $\operatorname{Arg} f'(1+i)$.

Now $f'(z) = -1/z^2$, so

$$f'(1+i) = -\frac{1}{(1+i)^2} = -\frac{1}{2i} = \frac{i}{2},$$

which has modulus $1/2$ and principal argument $\pi/2$.

So f scales the disc by the factor $1/2$ and rotates it anticlockwise through the angle $\pi/2$.

Exercise 1.10

Using the notion of a complex scale factor, describe what happens to points close to i under the function

$$f(z) = \frac{4z+3}{2z^2+1}.$$

It is important to bear in mind that the complex scale factor interpretation of a derivative is only an approximation, and that it is unlikely to be reliable far from the point under consideration. In the final section of this unit, we return to the scale factor interpretation and show how it can be described more precisely.

Further exercises

Exercise 1.11

Use the definition of derivative to find the derivative of the function

$$f(z) = 2z^2 + 5.$$

Exercise 1.12

Prove the Quotient Rule for differentiation.

Exercise 1.13

Find the derivative of each of the following functions f. In each case specify the domain of f'.

(a) $f(z) = \dfrac{z^2 + 2z + 1}{3z + 1}$ (b) $f(z) = \dfrac{z^3 + 1}{z^2 - z - 6}$

(c) $f(z) = \dfrac{1}{z^2 + 2z + 2}$ (d) $f(z) = z^2 + 5z - 2 + \dfrac{1}{z} + \dfrac{1}{z^2}$

Exercise 1.14

Use Strategy B to show that there are no points of $\mathbb{C}$ at which the function

$$f(z) = \operatorname{Im} z$$

is differentiable.

Exercise 1.15

Describe the approximate geometric effect of the function

$$f(z) = \frac{z^3 + 8}{z - 6}$$

on a small disc centred at the point 2.

2 The Cauchy–Riemann equations

After working through this section, you should be able to:

- find the *partial derivatives* of a function from $\mathbb{R}^2$ to $\mathbb{R}$
- use the *Cauchy–Riemann equations* to show that a function is *not* differentiable at a given point
- use the Cauchy–Riemann equations to show that a function, such as the exponential function, *is* differentiable at a given point, and to find the derivative.

This section is more challenging than some of the other sections, so you may find that you do not appreciate some of the details on a first reading. Most importantly, you should try to understand the definitions, strategies and theorems, and apply them in the examples and exercises.

2.1 The Cauchy–Riemann theorems

In this subsection we explore the relationship between complex differentiation and real differentiation. To do this, we introduce the notion of a *partial derivative* and use it to derive the *Cauchy–Riemann equations* (pronounced 'coh-she ree-man'). These equations are conditions that any differentiable complex function must satisfy, so they can be used to test whether a given complex function is differentiable. In particular, we use them to investigate the differentiability of the complex exponential function. The technique is to split the exponential function

$$\exp(x+iy) = e^x(\cos y + i\sin y)$$

into its real and imaginary parts:

$$u(x,y) = e^x \cos y \quad \text{and} \quad v(x,y) = e^x \sin y,$$

each of which is a real-valued function of the real variables x and y. The derivative of exp is then calculated by using the derivatives of the *real* trigonometric and exponential functions, which we assume to be known.

Before we deal with the exponential function, however, let us first consider the simpler function $f(z) = z^3$. By writing $z = x + iy$, we see that

$$f(x+iy) = (x+iy)^3 = (x^3 - 3xy^2) + i(3x^2y - y^3).$$

Let us define

$$u(x,y) = x^3 - 3xy^2 \quad \text{and} \quad v(x,y) = 3x^2y - y^3.$$

Then u and v are the real and imaginary parts of f, respectively; that is, $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$. For the moment we will concentrate on the real part u; part of its graph (given by the equation $s = u(x,y)$) is shown in Figure 2.1. Since u is a function of two real variables, its graph is a surface. The height of the surface above the (x,y)-plane represents the value of the function at the point (x,y). For instance, the point P on the surface has coordinates $(2,1,2)$ because $u(2,1) = 2^3 - 3\times 2\times 1^2 = 2$.

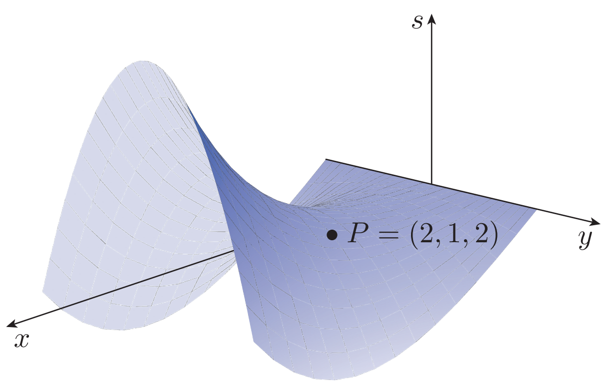


Figure 2.1 Graph of $u(x,y) = x^3 - 3xy^2$

Let us now explore the concept of the gradient of the surface at a point such as P. We will find that the answer depends on the 'direction' from which we approach the point. To make this more precise, consider Figure 2.2, in which the vertical plane with equation $y = 1$ is shown intersecting the surface in a curve that passes through P. By substituting $y = 1$ into $u(x, y) = x^3 - 3xy^2$, we see that the curve has equation $x \longmapsto x^3 - 3x$, so we can calculate its gradient at P; this is the gradient of the surface in the x-direction at P.

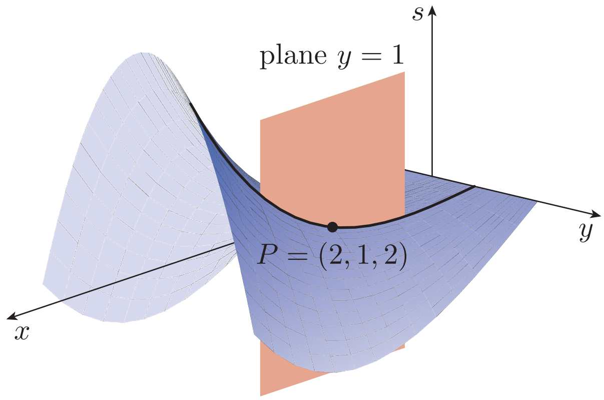


Figure 2.2 Intersection of the graph of $u(x, y) = x^3 - 3xy^2$ with the vertical plane $y = 1$

More generally, whenever we intersect the surface with a vertical plane with equation $y =$ constant, we obtain a curve on the surface with equation $x \longmapsto x^3 - 3xy^2$ (where y is considered to be fixed). We can find the gradient at any point $(a, b, u(a, b))$ on this curve by differentiating with respect to x and then substituting $x = a$ and $y = b$. The resulting expression is called the *partial derivative of u with respect to x at* (a, b), and it is denoted by

$$\frac{\partial u}{\partial x}(a, b).$$

A curly ∂ is used rather than a straight d to emphasise that this is a *partial* derivative, for which we differentiate with respect to one variable and keep the other variable fixed. In our particular case, differentiating $u(x, y) = x^3 - 3xy^2$ with respect to x (and keeping y fixed) gives

$$\frac{\partial u}{\partial x}(x, y) = 3x^2 - 3y^2,$$

and substituting $x = 2$ and $y = 1$ gives

$$\frac{\partial u}{\partial x}(2, 1) = 9.$$

Hence the gradient of the surface in the x-direction at the point P is 9. This is a positive value because near the point P, u increases as x increases (with $y = 1$), as you can see from Figure 2.2.

Figure 2.3 shows the vertical plane with equation $x = 2$ intersecting the surface in a different curve that passes through P.

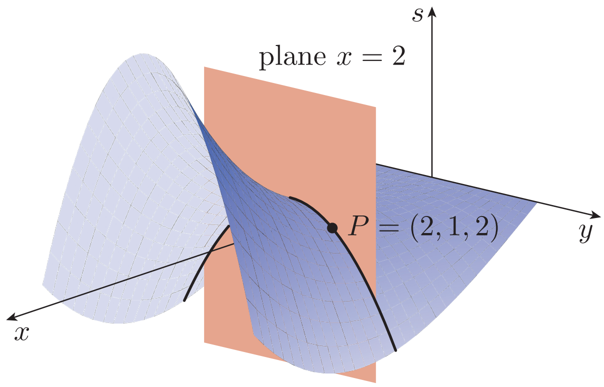


Figure 2.3 Intersection of the graph of $u(x, y) = x^3 - 3xy^2$ with the vertical plane $x = 2$

Reasoning similarly to before, we see that intersecting the surface with a vertical plane with equation $x =$ constant gives a curve on the surface, and we can obtain the gradient at a point $(a, b, u(a, b))$ on this curve by differentiating $u(x, y)$ with respect to y while keeping x fixed (and then substituting $x = a$ and $y = b$). The resulting expression is called the *partial derivative of u with respect to y at* (a, b), and it is denoted by

$$\frac{\partial u}{\partial y}(a, b).$$

Differentiating $u(x, y) = x^3 - 3xy^2$ with respect to y (and keeping x fixed) gives

$$\frac{\partial u}{\partial y}(x, y) = -6xy, \quad \text{so} \quad \frac{\partial u}{\partial y}(2, 1) = -12;$$

this is the gradient of the surface in the y-direction at the point P. It is a negative value this time, because when x and y are positive, u decreases as y increases (keeping x fixed), as you can see from Figure 2.3.

You will need to work with partial derivatives a good deal in this unit, so let us state the definitions formally.

Definitions

Let $u\colon A \longrightarrow \mathbb{R}$ be a function with domain A a subset of $\mathbb{R}^2$ that contains the point (a, b).

- The **partial derivative of u with respect to x at (a, b)**, denoted $\dfrac{\partial u}{\partial x}(a, b)$, is the derivative of the function $x \longmapsto u(x, b)$ at $x = a$, provided that this derivative exists.
- The **partial derivative of u with respect to y at (a, b)**, denoted $\dfrac{\partial u}{\partial y}(a, b)$, is the derivative of the function $y \longmapsto u(a, y)$ at $y = b$, provided that this derivative exists.

Partial derivatives are (at least in this module) *real* derivatives, not complex derivatives.

The next exercise asks you to work out the partial derivatives of the imaginary part of the complex function $f(z) = z^3$.

Exercise 2.1

(a) Calculate the partial derivatives of $v(x, y) = 3x^2y - y^3$.

(b) Evaluate these partial derivatives at $(2, 1)$.

Let us collect together the partial derivatives of the real and imaginary parts u and v of the function $f(z) = z^3$:

$$\frac{\partial u}{\partial x}(a,b) = 3a^2 - 3b^2, \qquad \frac{\partial v}{\partial x}(a,b) = 6ab,$$
$$\frac{\partial u}{\partial y}(a,b) = -6ab, \qquad \frac{\partial v}{\partial y}(a,b) = 3a^2 - 3b^2.$$

As you can see, we have

$$\frac{\partial u}{\partial x}(a,b) = \frac{\partial v}{\partial y}(a,b) \quad \text{and} \quad \frac{\partial v}{\partial x}(a,b) = -\frac{\partial u}{\partial y}(a,b).$$

This pair of equations is called the *Cauchy–Riemann equations*, and they hold true for the real and imaginary parts of any differentiable complex function, as the following important theorem testifies.

Theorem 2.1 Cauchy–Riemann Theorem

Let $f(x + iy) = u(x, y) + iv(x, y)$ be defined on a region $\mathcal{R}$ containing $a + ib$.

If f is differentiable at $a + ib$, then $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ exist at (a, b) and satisfy the **Cauchy–Riemann equations**

$$\frac{\partial u}{\partial x}(a,b) = \frac{\partial v}{\partial y}(a,b) \quad \text{and} \quad \frac{\partial v}{\partial x}(a,b) = -\frac{\partial u}{\partial y}(a,b).$$

Proof Let $\alpha = a + ib$. Suppose that (z_n) is *any* sequence in $\mathcal{R} - \{\alpha\}$ that converges to α. Let us write $z_n = x_n + iy_n$. According to the definition of a derivative, we have

$$f'(\alpha) = \lim_{z\to\alpha} \frac{f(z) - f(\alpha)}{z - \alpha} = \lim_{n\to\infty} \frac{f(z_n) - f(\alpha)}{z_n - \alpha}.$$

Observe that, by expressing f in terms of its real and imaginary parts, we can write

$$\frac{f(z_n) - f(\alpha)}{z_n - \alpha} = \left(\frac{u(x_n, y_n) - u(a,b)}{(x_n - a) + i(y_n - b)}\right) + i\left(\frac{v(x_n, y_n) - v(a,b)}{(x_n - a) + i(y_n - b)}\right). \tag{2.1}$$

We proceed by choosing two different types of sequences (z_n), and observing the behaviour of the expressions in large brackets in equation (2.1) in each case.

For our first choice, let us begin by defining (x_n) to be any sequence in $\mathbb{R} - \{a\}$ that converges to a. Let $z_n = x_n + ib$, so the sequence (z_n) converges to $\alpha = a + ib$. By removing a finite number of terms from (x_n), if need be, we can assume that each point z_n belongs to the open set $\mathcal{R} - \{\alpha\}$. Substituting $z_n = x_n + ib$ into equation (2.1) gives

$$\frac{f(z_n) - f(\alpha)}{z_n - \alpha} = \left(\frac{u(x_n, b) - u(a, b)}{x_n - a}\right) + i\left(\frac{v(x_n, b) - v(a, b)}{x_n - a}\right).$$

We know that the expression on the left-hand side converges (to $f'(\alpha)$), so its real and imaginary parts (indicated by the bracketed expressions on the right-hand side) converge too. Since (x_n) was chosen to be *any* sequence in $\mathbb{R} - \{a\}$ that converges to a, we see from the definition of partial derivatives that $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial v}{\partial x}$ exist at (a, b) and

$$\frac{u(x_n, b) - u(a, b)}{x_n - a} \to \frac{\partial u}{\partial x}(a, b) \quad \text{and} \quad \frac{v(x_n, b) - v(a, b)}{x_n - a} \to \frac{\partial v}{\partial x}(a, b).$$

In summary, we have

$$f'(\alpha) = \frac{\partial u}{\partial x}(a, b) + i\frac{\partial v}{\partial x}(a, b). \tag{2.2}$$

Next let (y_n) be any sequence in $\mathbb{R} - \{b\}$ that converges to b, and define $z_n = a + iy_n$, so $z_n \to \alpha$. Again, by omitting a finite number of terms from (y_n), if need be, we can assume that $z_n \in \mathcal{R} - \{\alpha\}$ for all n. Substituting $z_n = a + iy_n$ into equation (2.1) gives

$$\begin{aligned}\frac{f(z_n) - f(\alpha)}{z_n - \alpha} &= \left(\frac{u(a, y_n) - u(a, b)}{i(y_n - b)}\right) + i\left(\frac{v(a, y_n) - v(a, b)}{i(y_n - b)}\right)\\ &= \left(\frac{v(a, y_n) - v(a, b)}{y_n - b}\right) - i\left(\frac{u(a, y_n) - u(a, b)}{y_n - b}\right).\end{aligned}$$

Reasoning as before, we see that $\dfrac{\partial u}{\partial y}$ and $\dfrac{\partial v}{\partial y}$ exist at (a, b) and

$$f'(\alpha) = \frac{\partial v}{\partial y}(a, b) - i\frac{\partial u}{\partial y}(a, b). \tag{2.3}$$

Comparing equations (2.2) and (2.3), and equating real and imaginary parts, we obtain the Cauchy–Riemann equations, as required. ■

Origin of the Cauchy–Riemann equations

The Cauchy–Riemann equations are named after the mathematicians Augustin-Louis Cauchy (whom you encountered in Unit A3) and Bernhard Riemann (1826–1866), who were among the first to recognise the importance of these equations in complex analysis. We will meet these two mathematicians again in later units.

Jean le Rond d'Alembert

The Cauchy–Riemann equations first appeared in the work of another mathematician, however: the Frenchman Jean le Rond d'Alembert (1717–1783), who is perhaps best remembered for his work in classical mechanics. Indeed, the Cauchy–Riemann equations were written down by d'Alembert in an essay on fluid dynamics in 1752 to describe the velocity components of a two-dimensional irrotational fluid flow. You will learn about fluid flows in Unit D1.

The Cauchy–Riemann Theorem gives us another strategy for proving the non-differentiability of a complex function. (Two other strategies were described in Subsection 1.3.) If a complex function is differentiable, then it must satisfy the Cauchy–Riemann equations. So if those equations do not hold, then the function cannot be differentiable.

Strategy C for non-differentiability

Let $f(x+iy) = u(x,y) + iv(x,y)$. If either

$$\frac{\partial u}{\partial x}(a,b) \neq \frac{\partial v}{\partial y}(a,b) \quad \text{or} \quad \frac{\partial v}{\partial x}(a,b) \neq -\frac{\partial u}{\partial y}(a,b),$$

then f is not differentiable at $a+ib$.

To illustrate this strategy, consider the function

$$f(x+iy) = (x^2+y^2) + i(2x+4y).$$

The real part u and imaginary part v of this function are given by

$$u(x,y) = x^2+y^2 \quad \text{and} \quad v(x,y) = 2x+4y.$$

Hence

$$\frac{\partial u}{\partial x}(x,y) = 2x, \qquad \frac{\partial v}{\partial x}(x,y) = 2,$$

$$\frac{\partial u}{\partial y}(x,y) = 2y, \qquad \frac{\partial v}{\partial y}(x,y) = 4.$$

As you can see, the partial derivatives have been grouped into two pairs according to the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) \quad \text{and} \quad \frac{\partial v}{\partial x}(x,y) = -\frac{\partial u}{\partial y}(x,y).$$

In this case, these equations are $2x = 4$ and $2 = -2y$, which are satisfied only when $x = 2$ and $y = -1$; that is, they are satisfied only when $z = 2 - i$. If $z \neq 2 - i$, then the Cauchy–Riemann equations fail, so Strategy C tells us that f is not differentiable at z.

Notice that the Cauchy–Riemann Theorem and Strategy C do *not* tell us whether f is differentiable at the point $2-i$ at which the Cauchy–Riemann equations are satisfied. To deal with points of this type we need another theorem, which we will come to shortly. First, however, try the following exercise, to practise applying Strategy C.

Exercise 2.2

Show that each of the following functions fails to be differentiable at all points of $\mathbb{C}$.

(a) $f(x+iy) = e^x - ie^y$ (b) $f(z) = \overline{z}$

We have seen that if the Cauchy–Riemann equations are *not* satisfied, then the function is not differentiable. Let us now describe an example to show that even if the Cauchy–Riemann equations *are* satisfied, then the function may still not be differentiable.

Consider the function $f(x+iy) = u(x,y) + iv(x,y)$, where $v(x,y) = 0$ for all x and y, and

$$u(x,y) = \begin{cases} \min\{x,y\}, & x,y>0, \\ 0, & \text{otherwise.} \end{cases}$$

The graph of u is shown in Figure 2.4.

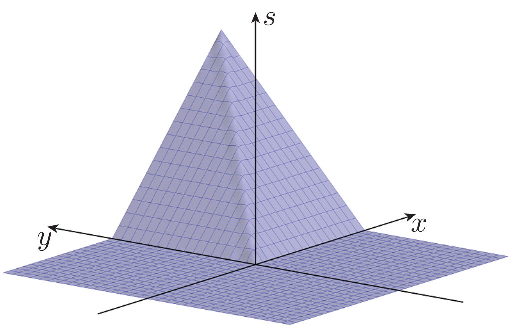


Figure 2.4 Graph of u

Since u and v take the value 0 at all points on the x- and y-axes, we see that all the partial derivatives vanish at $(0,0)$; that is,

$$\frac{\partial u}{\partial x}(0,0) = 0, \quad \frac{\partial v}{\partial x}(0,0) = 0,$$
$$\frac{\partial u}{\partial y}(0,0) = 0, \quad \frac{\partial v}{\partial y}(0,0) = 0.$$

However, even though the Cauchy–Riemann equations are satisfied at the origin, f is *not* differentiable there. To see this, observe that if $z_n = 1/n$, $n = 1, 2, \ldots$, then

$$\frac{f(z_n) - f(0)}{z_n - 0} = \frac{u(1/n, 0) - 0}{1/n - 0} = 0 \to 0,$$

whereas if $z_n = 1/n + i/n$, $n = 1, 2, \dots$, then

$$\frac{f(z_n) - f(0)}{z_n - 0} = \frac{u(1/n, 1/n) - 0}{1/n + i/n - 0} = \frac{1/n}{1/n + i/n} = \frac{1}{1+i} \to \frac{1}{1+i}.$$

The two limits 0 and $1/(1+i)$ differ, so f is not differentiable at 0.

This example demonstrates that the differentiability of a complex function does not follow from the Cauchy–Riemann equations alone. However, if certain extra conditions are satisfied, then f is differentiable, as the following theorem reveals.

Theorem 2.2 Cauchy–Riemann Converse Theorem

Let $f(x+iy) = u(x,y) + iv(x,y)$ be defined on a region $\mathcal{R}$ containing $a+ib$. If the partial derivatives $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$

- exist at (x,y) for each $x + iy \in \mathcal{R}$
- are continuous at (a,b)
- satisfy the Cauchy–Riemann equations at (a,b),

then f is differentiable at $a+ib$ and

$$f'(a+ib) = \frac{\partial u}{\partial x}(a,b) + i\frac{\partial v}{\partial x}(a,b).$$

The proof of this theorem is postponed until the next subsection.

Let us now return to the function $f(x+iy) = (x^2+y^2) + i(2x+4y)$, considered earlier, which satisfies the Cauchy–Riemann equations at the point $z = 2 - i$ only, and is therefore not differentiable at any other point. You saw earlier that the partial derivatives exist for every point (x,y) (so we can choose $\mathcal{R} = \mathbb{C}$ in applying Theorem 2.2) and they satisfy

$$\frac{\partial u}{\partial x}(x,y) = 2x, \quad \frac{\partial v}{\partial x}(x,y) = 2,$$
$$\frac{\partial u}{\partial y}(x,y) = 2y, \quad \frac{\partial v}{\partial y}(x,y) = 4.$$

Each of these functions is continuous at $(2,-1)$ because each of them is either constant or a multiple of one of the basic continuous functions $\operatorname{Re} z$ or $\operatorname{Im} z$. For example, the function $(x,y) \longmapsto 2x$ can be thought of as $z \longmapsto 2\operatorname{Re} z$.

It follows, then, from the Cauchy–Riemann Converse Theorem that f is differentiable at $2-i$. In fact, the theorem even tells us the value of $f'(2-i)$, namely

$$f'(2-i) = \frac{\partial u}{\partial x}(2,-1) + i\frac{\partial v}{\partial x}(2,-1) = 2 \times 2 + i \times 2 = 4 + 2i.$$

Finally in this subsection, we investigate the differentiability of the complex exponential function, as promised earlier.

Example 2.1

Prove that the complex exponential function $f(z) = e^z$ is entire, and find its derivative.

Solution

The real part u and the imaginary part v of f are given by

$$u(x, y) = e^x \cos y \quad \text{and} \quad v(x, y) = e^x \sin y.$$

Hence the partial derivatives of u and v exist for every point (x, y) and satisfy

$$\frac{\partial u}{\partial x}(x, y) = e^x \cos y, \qquad \frac{\partial v}{\partial x}(x, y) = e^x \sin y,$$
$$\frac{\partial u}{\partial y}(x, y) = -e^x \sin y, \quad \frac{\partial v}{\partial y}(x, y) = e^x \cos y.$$

Since the real exponential and trigonometric functions are continuous, and the real and imaginary part functions $\operatorname{Re} z$ and $\operatorname{Im} z$ are basic continuous functions, we see from the Combination Rules and Composition Rule for continuous functions that each partial derivative is continuous at every point (x, y).

The Cauchy–Riemann equations are satisfied at all points (x, y), so the Cauchy–Riemann Converse Theorem tells us that f is differentiable at every point of the complex plane (it is entire) and

$$f'(z) = \frac{\partial u}{\partial x}(x, y) + i\frac{\partial v}{\partial x}(x, y) = e^x \cos y + ie^x \sin y = e^z.$$

Exercise 2.3

Use the Cauchy–Riemann theorems to find the derivatives of the following functions. In each case specify the domain of the derivative.

(a) $f(z) = \sin z$ (b) $f(z) = |z|^2$

(*Hint*: For part (a), write $\sin z = \sin(x + iy)$ and use a trigonometric addition identity from Unit A2 to find the real and imaginary parts of $\sin z$.)

2.2 Proof of the Cauchy–Riemann Converse Theorem

The proof of the Cauchy–Riemann Converse Theorem is rather involved and may be omitted on a first reading.

We will need two results from real analysis. The first result is known as the Mean Value Theorem.

Theorem 2.3 Mean Value Theorem

Let f be a real function that is continuous on the closed interval $[a, x]$ and differentiable on the open interval (a, x). Then there is a number $c \in (a, x)$ such that

$$f(x) = f(a) + (x - a)f'(c). \tag{2.4}$$

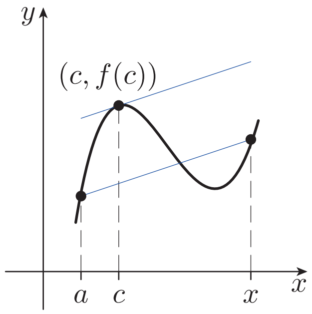


Figure 2.5 Graph of the real function f

To appreciate why this theorem is true, imagine pushing the chord between $(a, f(a))$ and $(x, f(x))$ in Figure 2.5 parallel to itself until it becomes a tangent to the graph of f at a point $(c, f(c))$, where c lies somewhere between a and x. Clearly, the gradient of the original chord must be equal to the gradient of the tangent, so

$$\frac{f(x) - f(a)}{x - a} = f'(c).$$

Multiplication by $x - a$ gives $f(x) = f(a) + (x - a)f'(c)$. Notice that this equation is also true if $x = c = a$.

The second result that we will need is a Linear Approximation Theorem, which asserts that if u is a real-valued function of two real variables x and y, then for (x, y) near (a, b), the value of $u(x, y)$ can be approximated by the value of the linear function t defined by

$$t(x, y) = u(a, b) + (x - a)\frac{\partial u}{\partial x}(a, b) + (y - b)\frac{\partial u}{\partial y}(a, b).$$

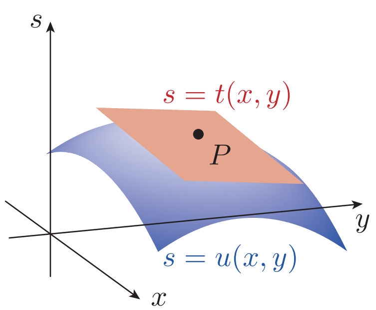


Figure 2.6 Tangent plane to the graph of u at the point P

Now, the graph of t is a plane passing through the point $P = (a, b, u(a, b))$ on the graph of u (Figure 2.6). Moreover, the partial x- and y-derivatives of t coincide with the partial x- and y-derivatives of u at (a, b). This means that both have the same gradient in the x- and y-directions, so you can think of the plane as the *tangent plane* to the graph of u at P.

The accuracy with which this tangent plane approximates the graph of u depends on the smoothness of the graph of u. If the graph exhibits the kind of kink shown in Figure 2.4, then the approximation is not as good as for a function with continuous partial derivatives.

Theorem 2.4 Linear Approximation Theorem ($\mathbb{R}^2$ to $\mathbb{R}$)

Let u be a real-valued function of two real variables, defined on a region $\mathcal{R}$ in $\mathbb{R}^2$ containing (a, b). If the partial x- and y-derivatives of u exist on $\mathcal{R}$ and are continuous at (a, b), then there is an 'error function' e such that

$$u(x, y) = u(a, b) + (x - a)\frac{\partial u}{\partial x}(a, b) + (y - b)\frac{\partial u}{\partial y}(a, b) + e(x, y),$$

where $\dfrac{e(x, y)}{\sqrt{(x - a)^2 + (y - b)^2}} \to 0$ as $(x, y) \to (a, b)$.

Since $\sqrt{(x-a)^2+(y-b)^2}$ is the distance from (a,b) to (x,y), the theorem asserts that the error function tends to zero 'faster' than this distance. Theorem 2.4 is the real-valued function analogue of Theorem 1.2.

Proof We have to show that the function e defined by

$$e(x,y)=u(x,y)-u(a,b)-(x-a)\frac{\partial u}{\partial x}(a,b)-(y-b)\frac{\partial u}{\partial y}(a,b)$$

satisfies

$$\frac{e(x,y)}{\sqrt{(x-a)^2+(y-b)^2}}\to 0 \text{ as } (x,y)\to(a,b).$$

Since the partial derivatives exist on $\mathcal{R}$, they must be defined on some disc centred at (a,b). Let us begin by finding an expression for $u(x,y)-u(a,b)$ on this disc. If we apply the Mean Value Theorem to the real functions $x\longmapsto u(x,y)$ (where y is kept constant) and $y\longmapsto u(a,y)$, then we obtain

$$u(x,y)=u(a,y)+(x-a)\frac{\partial u}{\partial x}(r,y),$$

where r is between a and x, and

$$u(a,y)=u(a,b)+(y-b)\frac{\partial u}{\partial y}(a,s),$$

where s is between b and y (see Figure 2.7). Hence

$$\begin{aligned}u(x,y)-u(a,b)&=(u(x,y)-u(a,y))+(u(a,y)-u(a,b))\\&=(x-a)\frac{\partial u}{\partial x}(r,y)+(y-b)\frac{\partial u}{\partial y}(a,s).\end{aligned}$$

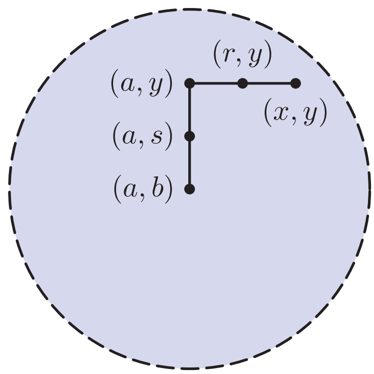


Figure 2.7 Vertical and horizontal line segments from (a,b) to (x,y)

Substituting this expression for $u(x,y)-u(a,b)$ into the definition of e, we obtain

$$e(x,y)=(x-a)\left(\frac{\partial u}{\partial x}(r,y)-\frac{\partial u}{\partial x}(a,b)\right)+(y-b)\left(\frac{\partial u}{\partial y}(a,s)-\frac{\partial u}{\partial y}(a,b)\right).$$

Dividing both sides by $\sqrt{(x-a)^2+(y-b)^2}$, and noting that

$$\frac{|x-a|}{\sqrt{(x-a)^2+(y-b)^2}}\leq 1 \quad\text{and}\quad \frac{|y-b|}{\sqrt{(x-a)^2+(y-b)^2}}\leq 1$$

(because both $(x-a)^2$ and $(y-b)^2$ do not exceed $(x-a)^2+(y-b)^2$), we see that

$$\left|\frac{e(x,y)}{\sqrt{(x-a)^2+(y-b)^2}}\right|\leq\left|\frac{\partial u}{\partial x}(r,y)-\frac{\partial u}{\partial x}(a,b)\right|+\left|\frac{\partial u}{\partial y}(a,s)-\frac{\partial u}{\partial y}(a,b)\right|.$$

Figure 2.7 illustrates that as (x,y) tends to (a,b), so do (a,s) and (r,y). So, by the continuity of the partial x- and y-derivatives at (a,b), the two terms on the right of the inequality above must both tend to 0 as (x,y) tends to (a,b). It follows that $e(x,y)/\sqrt{(x-a)^2+(y-b)^2}$ tends to 0 as (x,y) tends to (a,b). ■

We are now in a position to prove the Cauchy–Riemann Converse Theorem.

Theorem 2.2 Cauchy–Riemann Converse Theorem

Let $f(x+iy) = u(x,y) + iv(x,y)$ be defined on a region $\mathcal{R}$ containing $a+ib$. If the partial derivatives $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$

- exist at (x,y) for each $x+iy \in \mathcal{R}$
- are continuous at (a,b)
- satisfy the Cauchy–Riemann equations at (a,b),

then f is differentiable at $a+ib$ and

$$f'(a+ib) = \frac{\partial u}{\partial x}(a,b) + i\frac{\partial v}{\partial x}(a,b).$$

Proof We need to show that the limit of the difference quotient for f at $\alpha = a+ib$ exists and has the value indicated in the theorem. In order to calculate the difference quotient for f at α, we find an expression for $f(z) - f(\alpha)$. Since u and v fulfil the conditions of Theorem 2.4, it follows that

$$\begin{aligned} f(z) - f(\alpha) &= (u(x,y) - u(a,b)) + i(v(x,y) - v(a,b)) \\ &= \left((x-a)\frac{\partial u}{\partial x}(a,b) + (y-b)\frac{\partial u}{\partial y}(a,b) + e_u(x,y)\right) \\ &\quad + i\left((x-a)\frac{\partial v}{\partial x}(a,b) + (y-b)\frac{\partial v}{\partial y}(a,b) + e_v(x,y)\right), \end{aligned}$$

where e_u and e_v are the error functions associated with u and v, respectively.

Collecting together terms, we see that

$$\begin{aligned} f(z) - f(\alpha) &= (x-a)\left(\frac{\partial u}{\partial x}(a,b) + i\frac{\partial v}{\partial x}(a,b)\right) \\ &\quad + i(y-b)\left(\frac{\partial v}{\partial y}(a,b) - i\frac{\partial u}{\partial y}(a,b)\right) + e_u(x,y) + ie_v(x,y). \end{aligned}$$

Since u and v satisfy the Cauchy–Riemann equations, both expressions in the large brackets must be equal, so

$$\begin{aligned} f(z) - f(\alpha) &= ((x-a) + i(y-b))\left(\frac{\partial u}{\partial x}(a,b) + i\frac{\partial v}{\partial x}(a,b)\right) \\ &\quad + e_u(x,y) + ie_v(x,y). \end{aligned}$$

Dividing by $z - \alpha = (x-a) + i(y-b)$ gives

$$\frac{f(z) - f(\alpha)}{z - \alpha} = \left(\frac{\partial u}{\partial x}(a,b) + i\frac{\partial v}{\partial x}(a,b)\right) + \left(\frac{e_u(x,y) + ie_v(x,y)}{(x-a) + i(y-b)}\right).$$

The limit $f'(\alpha)$ of this difference quotient exists, and has the required value

$$\frac{\partial u}{\partial x}(a,b) + i\frac{\partial v}{\partial x}(a,b),$$

provided that we can show that the expression involving the

error functions e_u and e_v tends to 0 as $z = x + iy$ tends to α. To this end, notice that $|(x-a)+i(y-b)|$ is equal to $\sqrt{(x-a)^2+(y-b)^2}$ and so, by the Triangle Inequality,

$$\left|\frac{e_u(x,y)+ie_v(x,y)}{(x-a)+i(y-b)}\right| \leq \left|\frac{e_u(x,y)}{\sqrt{(x-a)^2+(y-b)^2}}\right| + \left|\frac{e_v(x,y)}{\sqrt{(x-a)^2+(y-b)^2}}\right|.$$

By Theorem 2.4, both expressions on the right tend to 0 as $x + iy$ tends to α. Consequently, the expression on the left must also tend to 0, and the theorem follows. ■

Further exercises

Exercise 2.4

Calculate the partial derivatives $\dfrac{\partial u}{\partial x}(x,y)$ and $\dfrac{\partial u}{\partial y}(x,y)$ of each of the following functions.

(a) $u(x,y) = 3x + xy + 2x^2y^2$ (b) $u(x,y) = x\cos y + \exp(xy)$

(c) $u(x,y) = (x+y)^3$

Exercise 2.5

Calculate the partial derivatives $\dfrac{\partial u}{\partial x}(x,y)$ and $\dfrac{\partial u}{\partial y}(x,y)$ of each of the following functions, and evaluate these partial derivatives at $(1,0)$.

(a) $u(x,y) = x^3y - y\cos y$ (b) $u(x,y) = ye^x - xy^3$

Exercise 2.6

Find the gradient of the graph of $u(x,y) = x^2 + 2xy$ at the point $(1,2,5)$ in the x-direction and in the y-direction.

Exercise 2.7

Use the Cauchy–Riemann equations to show that there is no point of $\mathbb{C}$ at which the function

$$f(x+iy) = e^x(\sin y + i\cos y)$$

is differentiable.

Exercise 2.8

Use the Cauchy–Riemann equations to show that the function

$$f(x+iy) = (x^2+x-y^2) + i(2xy+y)$$

is entire, and find its derivative.

Exercise 2.9

Use the Cauchy–Riemann equations to find all the points at which the following functions are differentiable, and calculate their derivatives.

(a) $f(x+iy) = (x^2+y^2) + i(x^2-y^2)$ (b) $f(x+iy) = xy$

Laplace's equation and electrostatics

The Cauchy–Riemann equations for a differentiable function $f(x+iy) = u(x,y) + iv(x,y)$ tell us that

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) \quad \text{and} \quad \frac{\partial v}{\partial x}(x,y) = -\frac{\partial u}{\partial y}(x,y).$$

These partial derivatives are themselves functions of x and y, so, provided that they are suitably well behaved, we can partially differentiate both sides of the first of the two equations with respect to x, and partially differentiate both sides of the second equation with respect to y, to obtain

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \text{and} \quad \frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2}.$$

(Here we have omitted the variables (x,y) after each derivative, for simplicity.) For sufficiently well-behaved functions, the two partial derivatives

$$\frac{\partial^2 v}{\partial x \partial y} \quad \text{and} \quad \frac{\partial^2 v}{\partial y \partial x}$$

are equal; the order in which you partially differentiate with respect to x and y does not matter. Hence

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2},$$

which implies that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

This equation for u is called **Laplace's equation**. (The imaginary part v of f satisfies Laplace's equation too.) It is named after the distinguished French mathematician and scientist Pierre-Simon Laplace (1749–1827), who studied the equation in his work on gravitational potentials.

Pierre-Simon Laplace

Laplace's equation has proved to have huge importance to physics. In a later unit, you will learn about its significance in fluid mechanics. It also has a key role in the subject of *electrostatics*. In that theory, it is known that the electrostatic potential $V(x,y)$ at a point (x,y) of a

region without charge satisfies Laplace's equation. It can be shown that V is the real part of some differentiable function f. Using these observations allows one to move between complex analysis and electrostatics: many of the theorems of complex analysis have important physical interpretations in electrostatics.

3 Rules for manipulating differentiable functions

After working through this section, you should be able to:

- use the Chain Rule, the Inverse Function Rule and the Restriction Rule to differentiate complex functions
- differentiate the principal logarithm and power functions
- use the table of standard derivatives.

Up to now we have described how to differentiate the complex exponential function and rational functions. However, this still leaves a large class of functions, such as $f(z) = \exp\left(1/z^2\right)$ and $f(z) = \operatorname{Log} z$. This section is about rules for differentiating composites and inverses.

3.1 The Chain Rule

In this subsection we introduce another useful technique for differentiating complex functions called the Chain Rule. If you have met the rule before for differentiating real functions, then you should find the technique and some of the examples familiar.

Using the geometric interpretation of complex differentiation discussed in Subsection 1.5, we would expect the composite $g \circ f$ of two differentiable functions to behave in the manner illustrated in Figure 3.1.

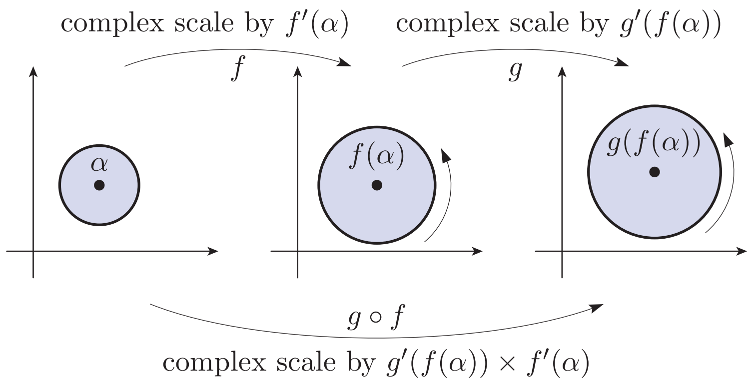


Figure 3.1 Composition of differentiable functions

To a close approximation, f rotates and scales any small disc centred at α by the complex scale factor $f'(\alpha)$, producing a small disc centred at $f(\alpha)$.

This is then rotated and scaled under g by a further factor $g'(f(\alpha))$, to produce a small disc centred at $g(f(\alpha))$. (Here we are assuming that both $f'(\alpha) \neq 0$ and $g'(f(\alpha)) \neq 0$.)

Overall, the original disc is scaled and rotated by the product of the two factors $g'(f(\alpha))$ and $f'(\alpha)$. This suggests that, as for real differentiation,

$$(g \circ f)'(\alpha) = g'(f(\alpha)) \times f'(\alpha).$$

The following theorem confirms that this is indeed the case. This theorem can be referred to as the Composition Rule (for differentiation), but it is more commonly called the Chain Rule.

Theorem 3.1 Chain Rule

Let f and g be complex functions, and let α be a limit point of the domain of $g \circ f$. If f is differentiable at α, and g is differentiable at $f(\alpha)$, then $g \circ f$ is differentiable at α, and

$$(g \circ f)'(\alpha) = g'(f(\alpha))f'(\alpha).$$

Remark

The assumptions that f is differentiable at α and g is differentiable at $f(\alpha)$ necessitate that α belongs to the domain of f and $f(\alpha)$ belongs to the domain of g. Consequently, α is a point in the domain of $g \circ f$. The Chain Rule also assumes that α is a *limit point* in the domain of $g \circ f$. This assumption is necessary in order for the limit

$$(g \circ f)'(\alpha) = \lim_{z \to \alpha} \frac{(g \circ f)(z) - (g \circ f)(\alpha)}{z - \alpha}$$

to be defined.

Before proving the Chain Rule we illustrate how it is used.

Example 3.1

Show that the function $k(z) = \exp\left(1/z^2\right)$ is analytic on its domain, and find its derivative.

Solution

The function k can be expressed as the composite $g \circ f$ of the functions

$$g(z) = \exp z \quad \text{and} \quad f(z) = 1/z^2.$$

The domain of $k = g \circ f$ is the set $A = \mathbb{C} - \{0\}$. Since this is a region, every point $z \in A$ is a limit point of A.

Now, f is differentiable on A, and g is differentiable on $\mathbb{C}$ and hence on $f(A)$. Furthermore,

$$g'(z) = \exp z \quad \text{and} \quad f'(z) = -2/z^3.$$

By the Chain Rule, $k = g \circ f$ is differentiable on A, and hence it is analytic on A, since A is region. The Chain Rule gives

$$\begin{aligned} k'(z) &= g'(f(z))f'(z) \\ &= \exp\left(1/z^2\right) \times \left(-2/z^3\right) \\ &= -\frac{2\exp\left(1/z^2\right)}{z^3}. \end{aligned}$$

Exercise 3.1

Show that each of the following functions is analytic on its domain, and find its derivative.

(a) $k(z) = (z+5)^{900}$ (b) $k(z) = \exp(z^2+4)$

(c) $k(z) = e^{\alpha z}$, where $\alpha \in \mathbb{C}$

The next example shows how part (c) of Exercise 3.1 can be used to differentiate the standard trigonometric functions.

Example 3.2

Show that each of the following complex functions is analytic on its domain, and find its derivative.

(a) sin (b) cos (c) tan

Solution

Since the functions $z \longmapsto e^{iz}$ and $z \longmapsto e^{-iz}$ are entire, with derivatives $z \longmapsto ie^{iz}$ and $z \longmapsto -ie^{-iz}$, respectively, it follows from the Combination Rules for differentiation that:

(a) $\sin z = \left(e^{iz} - e^{-iz}\right)/2i$ is entire, and

$$\sin' z = \frac{ie^{iz} - \left(-ie^{-iz}\right)}{2i} = \frac{e^{iz} + e^{-iz}}{2} = \cos z$$

(b) $\cos z = \left(e^{iz} + e^{-iz}\right)/2$ is entire, and

$$\cos' z = \frac{ie^{iz} + \left(-ie^{-iz}\right)}{2} = \frac{-\left(e^{iz} - e^{-iz}\right)}{2i} = -\sin z$$

(c) $\tan z = \sin z / \cos z$ is analytic on its domain $\mathbb{C} - \{(n+\frac{1}{2})\pi : n \in \mathbb{Z}\}$, and

$$\tan' z = \frac{\cos z \times \cos z - \sin z \times (-\sin z)}{\cos^2 z} = \frac{1}{\cos^2 z} = \sec^2 z.$$

The hyperbolic functions can be differentiated in a similar way.

Exercise 3.2

Show that each of the following complex functions is analytic on its domain, and find its derivative.

(a) sinh (b) cosh (c) tanh

The Chain Rule can be extended to composites of three (or more) functions. For example, if α is a limit point of the domain of the composite $h \circ g \circ f$, and if h, g and f are differentiable at $g(f(\alpha))$, $f(\alpha)$ and α, respectively, then

$$\begin{aligned}(h \circ g \circ f)'(\alpha) &= (h \circ g)'(f(\alpha)) \times f'(\alpha) \\ &= h'(g(f(\alpha))) \times g'(f(\alpha)) \times f'(\alpha).\end{aligned}$$

In general, to differentiate the composite of two or more functions, simply take the product of all the derivatives, keeping the points at which the derivatives of the functions are evaluated the same as they were prior to differentiation. With a little practice, you should be able to find the derivative of a composite function without explicitly writing down the intermediate functions.

Example 3.3

Write down the derivative of the function $k(z) = \sin(\cosh(\tan z))$. Explain why the derivative has the same domain as k.

Solution

Since k is a composite of the functions sin, cosh and tan, we can use the Chain Rule to write down the derivative of k:

$$k'(z) = \overbrace{\cos(\cosh(\tan z))}^{\text{derivative of sin}} \times \overbrace{\sinh(\tan z)}^{\text{derivative of cosh}} \times \overbrace{\sec^2 z}^{\text{derivative of tan}}.$$

To show that k' has the same domain as k, let α be any point in the domain of k, which is the set $E = \mathbb{C} - \{(n + \frac{1}{2})\pi : n \in \mathbb{Z}\}$. Since E is a region, every point α of E is a limit point of E. Furthermore, tan, cosh and sin are differentiable on the whole of their domains. So, in particular, they must be differentiable at α, $\tan \alpha$ and $\cosh(\tan \alpha)$, respectively. It follows that the Chain Rule can be applied throughout the domain of k, and that k' has the same domain as k.

Exercise 3.3

Show that each of the following functions k is analytic on its domain, and find its derivative.

(a) Use the formula for differentiating a composite of three functions, given above.

(i) $k(z) = (\sin^2 z + 3)^2$ (ii) $k(z) = \sin(\exp(\cos z - z))$

(b) Try to differentiate the composite functions by inspection – that is, without explicitly writing down the intermediate functions.

(i) $k(z) = \sin(\cosh z)$ (ii) $k(z) = \cos\big((1+z)^{20}\big)$

(iii) $k(z) = \exp(\exp(\sin z))$

Proof of the Chain Rule The idea is to write the difference quotient for $g \circ f$ in the form

$$\frac{g(f(z)) - g(f(\alpha))}{z-\alpha} = \left(\frac{g(w)-g(\beta)}{w-\beta}\right)\left(\frac{f(z)-f(\alpha)}{z-\alpha}\right), \tag{3.1}$$

where $w = f(z)$ and $\beta = f(\alpha)$, and then to let $z \to \alpha$ (and hence $w \to \beta$) to obtain

$$(g \circ f)'(\alpha) = g'(\beta) f'(\alpha),$$

as required. Unfortunately, there is a snag. Equation (3.1) does not make sense if $w = \beta$ (that is, if $f(z) = f(\alpha)$) and this may conceivably happen for values of z close to α. To avoid this problem, we introduce a function h with the same domain as g and rule

$$h(w) = \begin{cases} \dfrac{g(w)-g(\beta)}{w-\beta}, & w \neq \beta, \\ g'(\beta), & w = \beta. \end{cases}$$

Since $g'(\beta) = \lim\limits_{w\to\beta}((g(w)-g(\beta))/(w-\beta))$, this function h is continuous at β, by Theorem 3.1 of Unit A3. Now note that the equation

$$\frac{g(f(z)) - g(f(\alpha))}{z-\alpha} = h(f(z))\left(\frac{f(z)-f(\alpha)}{z-\alpha}\right)$$

holds for *all* z ($\neq \alpha$) in the domain of f (even if $f(z) = f(\alpha)$, since both sides then vanish). Hence, by the Product Rule for limits of functions,

$$\begin{aligned} \lim_{z\to\alpha} \frac{g(f(z)) - g(f(\alpha))}{z-\alpha} &= \lim_{z\to\alpha} h(f(z)) \lim_{z\to\alpha} \frac{f(z)-f(\alpha)}{z-\alpha} \\ &= h(f(\alpha)) f'(\alpha) \\ &= h(\beta) f'(\alpha) \\ &= g'(\beta) f'(\alpha), \end{aligned}$$

where, in the second line, we used the fact that $h \circ f$ is continuous at α, which is true because f is continuous at α (since it is differentiable at α) and h is continuous at $\beta = f(\alpha)$. Hence $(g \circ f)'(\alpha) = g'(f(\alpha)) f'(\alpha)$, as required. ∎

3.2 The Inverse Function Rule

One striking example of a function that cannot be differentiated using the rules that we have discussed so far is the function Log. Recall that Log is the inverse of the function exp when exp is restricted to the strip $\{z : -\pi < \operatorname{Im} z \leq \pi\}$ (see Subsection 5.1 of Unit A2). In this subsection we will show how Log can be differentiated, using a rule that relates the derivative of a function to the derivative of its inverse.

The relationship between a one-to-one function f and its inverse f^{-1} is illustrated in Figure 3.2. If f is differentiable at the point α and $f'(\alpha) \neq 0$, then, to a close approximation, a small disc in the domain, centred at α, is mapped to a small disc in the codomain, centred at $\beta = f(\alpha)$. In the process, the disc in the domain is rotated and scaled by the complex scale factor $f'(\alpha)$.

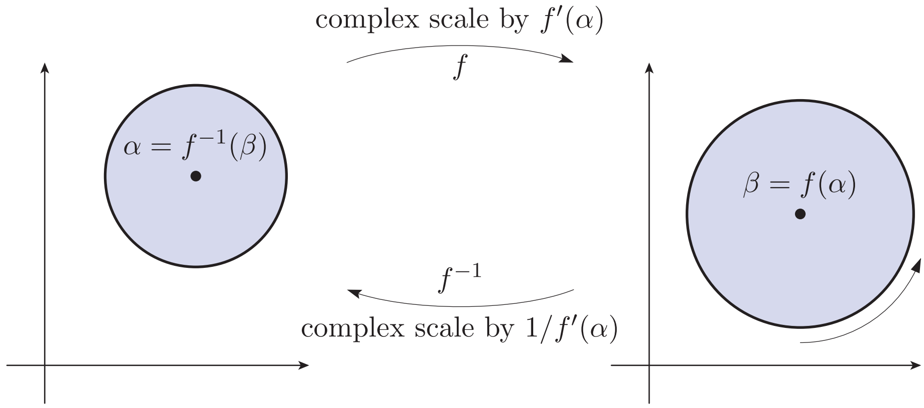


Figure 3.2 Complex scalings of a function and its inverse function

Now look at the process in reverse. The function f^{-1} takes the disc on the right, centred at β, to the disc on the left, centred at $\alpha = f^{-1}(\beta)$. As it does so, in order to undo the effect of f, the function f^{-1} rotates and scales the disc by the complex scale factor $1/f'(\alpha)$. If f^{-1} is differentiable at β, then this geometric reasoning suggests that

$$\left(f^{-1}\right)'(\beta) = 1/f'(\alpha) = 1/f'\left(f^{-1}(\beta)\right).$$

Theorem 3.2 Inverse Function Rule

Let $f\colon A \longrightarrow B$ be a one-to-one complex function, and suppose that f^{-1} is continuous at $\beta \in B$. If f has a non-zero derivative at $f^{-1}(\beta) \in A$, then f^{-1} is differentiable at β and

$$\left(f^{-1}\right)'(\beta) = \frac{1}{f'(f^{-1}(\beta))}.$$

We give a proof of the Inverse Function Rule at the end of this subsection, but first we illustrate how it is used.

At first sight, the rule may appear to be of limited use since many complex functions are not one-to-one. But, as we mentioned in Subsection 1.5 of Unit A2, it is usually possible to restrict the domain of a function in order to yield a new function that *is* one-to-one. For example, exp is not a one-to-one function, and yet the restriction defined by

$$f(z) = \exp z \quad (-\pi < \operatorname{Im} z \leq \pi),$$

is a one-to-one function whose inverse f^{-1} is the principal logarithm Log (with domain $\mathbb{C} - \{0\}$).

Note that we have used an abbreviated form for the domain of f in the formula above because the full version ($z \in \{z : \ldots\}$) is rather clumsy here. We will use such abbreviated forms when convenient.

Example 3.4

Find the derivative of Log.

Solution

Let f be the restriction of exp defined by

$$f(z) = \exp z \quad (-\pi < \operatorname{Im} z \leq \pi).$$

Then f is one-to-one and $f'(z) = \exp' z = \exp z \neq 0$.

At points on the negative real axis, $f^{-1} = \text{Log}$ is not differentiable because it is discontinuous. But at any other point in its domain, Log is continuous. So, by the Inverse Function Rule, the derivative of Log is

$$\text{Log}' z = \frac{1}{f'(f^{-1}(z))} = \frac{1}{\exp'(\text{Log}\, z)} = \frac{1}{\exp(\text{Log}\, z)} = \frac{1}{z},$$

where $z \in \mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$ (see Figure 3.3).

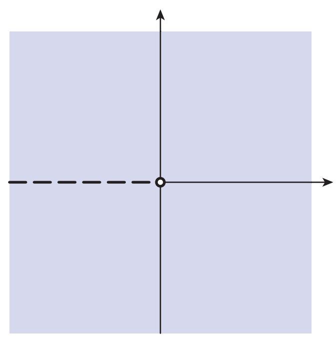


Figure 3.3 The cut plane $\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$

The derivative of Log is so important that we quote the result obtained in Example 3.4 as a corollary to the Inverse Function Rule.

Corollary

The derivative of the principal logarithm Log is

$$\text{Log}' z = \frac{1}{z} \quad (z \in \mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}).$$

Exercise 3.4

Find the derivative of the function $f(z) = \text{Log}(1 + iz)$, and specify its domain.

Having found the derivative of Log, we can now differentiate the principal power function $f(z) = z^\alpha$, by using the Chain Rule. Indeed, for any $\alpha \in \mathbb{C} - \mathbb{Z}$, we have

$$f(z) = z^\alpha = \exp(\alpha \operatorname{Log} z) \quad (z \in \mathbb{C} - \{0\}).$$

If $z \in \mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$, then, by the Chain Rule,

$$\begin{aligned} f'(z) &= \exp(\alpha \operatorname{Log} z) \times (\alpha \operatorname{Log}' z) \\ &= z^\alpha \times \alpha/z \\ &= \alpha z^{\alpha-1}. \end{aligned}$$

We state this result as another corollary to the Inverse Function Rule.

Corollary

Let $\alpha \in \mathbb{C} - \mathbb{Z}$. Then the derivative of the principal power function $f(z) = z^\alpha$ is

$$f'(z) = \alpha z^{\alpha-1} \quad (z \in \mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}).$$

Notice that this formula for differentiating *principal* powers is the same as the formula for differentiating *integer* powers given in Subsection 1.2. The only difference is the domain. For positive integer powers, the domain is $\mathbb{C}$; for negative integer powers, the domain is $\mathbb{C} - \{0\}$; and for general (principal) powers, the domain is the cut plane $\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$.

Exercise 3.5

Find the derivative of each of the following power functions. In each case specify the domain of the derivative.

(a) $f(z) = z^\pi$ (b) $f(z) = z^{3/2}$ (c) $f(z) = z^5$ (d) $f(z) = z^{-3}$

Proof of the Inverse Function Rule Let $f \colon A \longrightarrow B$ be a one-to-one complex function such that f^{-1} is continuous at $\beta \in B$, and suppose that f has a non-zero derivative at $f^{-1}(\beta) \in A$. We have to show that f^{-1} is differentiable at β and that

$$\left(f^{-1}\right)'(\beta) = \frac{1}{f'(f^{-1}(\beta))}.$$

We first check that β is a limit point of B. We do this by showing that there is a sequence in $B - \{\beta\}$ that converges to β. Since f is differentiable at $\alpha = f^{-1}(\beta)$, it follows that α is a limit point of A. In other words, α is the limit of a sequence (z_n) in $A - \{\alpha\}$. Now f is one-to-one, so $f(z_n) \neq f(\alpha)$, and $f(\alpha) = \beta$, so $(f(z_n))$ is a sequence in $B - \{\beta\}$. Furthermore, f is differentiable and hence continuous at α, so $(f(z_n))$ converges to β.

Next let (w_n) be any sequence in $B - \{\beta\}$ that converges to β. Since f^{-1} is continuous at β, the sequence (z_n), defined by $z_n = f^{-1}(w_n)$, must

converge to $\alpha = f^{-1}(\beta)$. Furthermore, f is one-to-one, so $z_n \neq \alpha$, since $w_n \neq \beta$. Hence

$$\begin{aligned}\lim_{n\to\infty} \frac{f^{-1}(w_n) - f^{-1}(\beta)}{w_n - \beta} &= \lim_{n\to\infty} \frac{z_n - \alpha}{f(z_n) - f(\alpha)} \\ &= \lim_{n\to\infty} \left(\frac{f(z_n) - f(\alpha)}{z_n - \alpha}\right)^{-1}.\end{aligned}$$

Since f is differentiable at α, and since $f'(\alpha) = f'(f^{-1}(\beta))$ is assumed to be non-zero, it follows from the Quotient Rule for sequences that the last limit exists, and is equal to $\dfrac{1}{f'(f^{-1}(\beta))}$, as required. ■

3.3 The Restriction Rule for Differentiation

When applying the Inverse Function Rule to the exponential function in the previous subsection, we ignored one technicality. We assumed that because the exponential function is differentiable, its restriction to the set

$$\{z : -\pi < \operatorname{Im} z \leq \pi\}$$

is also differentiable. This is not quite the same as saying that exp is differentiable on the set $\{z : -\pi < \operatorname{Im} z \leq \pi\}$. It is saying that a *new* function formed by restricting exp to $\{z : -\pi < \operatorname{Im} z \leq \pi\}$ is differentiable on its own domain.

The justification for assuming that the restriction of exp to $\{z : -\pi < \operatorname{Im} z \leq \pi\}$ is differentiable is provided by the following theorem.

Theorem 3.3 Restriction Rule for Differentiation

Let f and g be complex functions with domains A and B, respectively, and let $A \subseteq B$. If $\alpha \in A$ is a limit point of A and

- $f(z) = g(z)$, for $z \in A$
- g is differentiable at α,

then f is differentiable at α, and $f'(\alpha) = g'(\alpha)$.

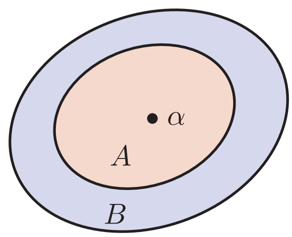


Figure 3.4 A point α in a set A contained in a set B

The domains A and B of f and g are illustrated in Figure 3.4.

Proof Let (z_n) be any sequence in $A - \{\alpha\}$ that converges to α. We need to show that

$$\lim_{n\to\infty} \frac{f(z_n) - f(\alpha)}{z_n - \alpha} = g'(\alpha).$$

Since $A \subseteq B$, (z_n) is a sequence in $B - \{\alpha\}$ that converges to α. It follows from the assumption that g is differentiable at α that

$$\lim_{n\to\infty} \frac{g(z_n) - g(\alpha)}{z_n - \alpha} = g'(\alpha).$$

Since $f(z) = g(z)$, for $z \in A$, we see that $f(z_n) = g(z_n)$ and $f(\alpha) = g(\alpha)$, so the result follows. ■

In essence, the Restriction Rule tells us that we need not worry too much about the domain of a function that we are trying to differentiate. It tells us that we can cut away unwanted parts of the domain of a differentiable function, provided that we avoid leaving 'isolated points', without affecting its differentiability at the remaining points.

In Unit A3 we learned that continuity is a 'local' property, in the sense that continuity of a function at a point depends only on the values of that function in any open disc (no matter how small) centred at the point. Differentiability is also a local property, for the same reasons, and it is because of this property that we can restrict the domain of a function while retaining differentiability.

The Restriction Rule explains why many real functions can be differentiated in the same way as their complex counterparts. For example, the derivative of the real sine function is the real cosine function; this can be explained by restricting the complex sine function to the real axis and applying the Restriction Rule. Similarly, the derivative of the real logarithm function can be obtained from the derivative of Log by restricting Log to the positive real axis $(0, \infty)$ and applying the Restriction Rule.

3.4 Standard derivatives

To end this section, we collect together a list of the functions that have been differentiated in this unit, together with their derivatives.

$f(z)$	$f'(z)$	Domain of f'
$\alpha, \quad \alpha \in \mathbb{C}$	0	$\mathbb{C}$
$z^k, \quad k \in \mathbb{Z},\ k > 0$	kz^{k-1}	$\mathbb{C}$
$z^k, \quad k \in \mathbb{Z},\ k < 0$	kz^{k-1}	$\mathbb{C} - \{0\}$
$z^\alpha, \quad \alpha \in \mathbb{C} - \mathbb{Z}$	$\alpha z^{\alpha-1}$	$\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$
$\exp z$	$\exp z$	$\mathbb{C}$
$\operatorname{Log} z$	$1/z$	$\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$
$\sin z$	$\cos z$	$\mathbb{C}$
$\cos z$	$-\sin z$	$\mathbb{C}$
$\tan z$	$\sec^2 z$	$\mathbb{C} - \left\{\left(n + \frac{1}{2}\right)\pi : n \in \mathbb{Z}\right\}$
$\sinh z$	$\cosh z$	$\mathbb{C}$
$\cosh z$	$\sinh z$	$\mathbb{C}$
$\tanh z$	$\operatorname{sech}^2 z$	$\mathbb{C} - \left\{\left(n + \frac{1}{2}\right)\pi i : n \in \mathbb{Z}\right\}$

Notice that the formula for differentiating non-zero powers always has the same form, but the domain of the derivative changes. For positive integer powers, the domain is $\mathbb{C}$; for negative integer powers, 0 is excluded; and for general (principal) powers, the negative reals and 0 are excluded.

By applying the Combination Rules, the Chain Rule and the Inverse Function Rule to this list of standard functions, it is possible to

differentiate most of the functions that you will encounter in this module. If necessary, the Restriction Rule can then be applied to obtain a function with the required domain.

Example 3.5

Find the derivatives of the following functions.

(a) $f(z) = 5 + 2\sin z \cosh z$ (b) $f(z) = \dfrac{e^z}{z + \sinh z - 2\cosh z}$

(c) $f(z) = z\exp\left(2z^2 + z\right)$

Solution

(a) By the Combination Rules,

$$f'(z) = 2(\cos z \cosh z + \sin z \sinh z).$$

(b) By the Combination Rules,

$$\begin{aligned} f'(z) &= \frac{(z + \sinh z - 2\cosh z)e^z - e^z(1 + \cosh z - 2\sinh z)}{(z + \sinh z - 2\cosh z)^2} \\ &= \frac{(-1 + z + 3\sinh z - 3\cosh z)e^z}{(z + \sinh z - 2\cosh z)^2}. \end{aligned}$$

(c) By the Chain Rule, the Product Rule and the rule for differentiating polynomials,

$$\begin{aligned} f'(z) &= 1 \times \exp\left(2z^2 + z\right) + z\left(\exp\left(2z^2 + z\right)\right)(4z + 1) \\ &= \left(1 + z + 4z^2\right)\exp\left(2z^2 + z\right). \end{aligned}$$

For further practice in using the rules of differentiation, try the following exercise.

Exercise 3.6

Find the derivatives of the following functions. In each case specify the domain of the derivative.

(a) $f(z) = 3 + e^z \operatorname{Log} z$ (b) $f(z) = (z + \sin z)^{20}$

(c) $f(z) = \cos^2(z + \cosh z)$ (d) $f(z) = \dfrac{\operatorname{Log} z}{z}$

Further exercises

Exercise 3.7

Find the rules of the derivatives of the following functions. (You are not required to find their domains.)

(a) $\operatorname{sech} z = \dfrac{1}{\cosh z}$ (b) $\operatorname{cosech} z = \dfrac{1}{\sinh z}$ (c) $\coth z = \dfrac{\cosh z}{\sinh z}$

Exercise 3.8

Use the rules of differentiation to find the derivative of the function

$$f(z) = \operatorname{Log}(z+1) + \exp(z^2).$$

Specify the domain of f'.

Exercise 3.9

Write down (without justification) the rules of the derivatives of the following functions.

(a) $f(z) = \sin(\cos z - z)$ (b) $f(z) = z\exp(z^2 + i) + \cos z$

(c) $f(z) = \dfrac{\operatorname{Log} z}{\exp(z+1)}$ (d) $f(z) = \cos(3\sin(i + \tan z))$

4 Smooth paths

After working through this section, you should be able to:

- differentiate a parametrisation
- decide whether a path is *smooth*
- interpret the derivative of an analytic function as a rotation and a scaling of tangent vectors
- find the *angle* between two smooth paths at a point of intersection
- understand the manner in which analytic functions preserve angles.

4.1 Derivatives of parametrisations

Recall that in Unit A2 we defined a *path* to be a subset Γ of $\mathbb{C}$ that is the image set of an associated continuous function $\gamma\colon I \longrightarrow \mathbb{C}$, where I is a real interval. The function γ is called a *parametrisation* (see Figure 4.1).

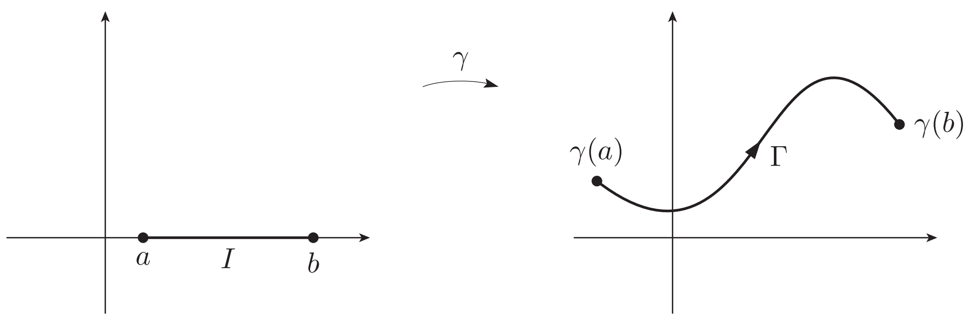


Figure 4.1 A path Γ with parametrisation $\gamma\colon I \longrightarrow \mathbb{C}$

We have drawn the interval I in Figure 4.1 as a subset of the complex plane to emphasise that γ is a complex function to which the theory of complex differentiation can be applied.

Exercise 4.1

Sketch the paths with the following parametrisations, using an arrow to indicate the direction of the path.

(a) $\gamma(t) = e^{it} \quad (t \in [0, 2\pi])$

(b) $\gamma(t) = \sin t + i|t| \quad (t \in [-\pi/2, \pi/2])$

In this section we apply the theory of differentiation to parametrisations. Geometrically, if the derivative of a parametrisation γ exists at a point c in I (and is non-zero), then it can be interpreted as a tangent vector to the path Γ at $\gamma(c)$. This is because the difference quotient

$$\frac{\gamma(t) - \gamma(c)}{t - c}$$

can be represented in the complex plane by a vector lying along the line through $\gamma(c)$ and $\gamma(t)$ (see Figure 4.2). In fact, it is the vector obtained by scaling the vector from $\gamma(c)$ to $\gamma(t)$ by the real factor $1/(t - c)$. Now, as t approaches c, the line through $\gamma(c)$ and $\gamma(t)$ becomes tangential to the path; so, in the limit, $\gamma'(c)$ can be represented as a tangent vector to the path at $\gamma(c)$.

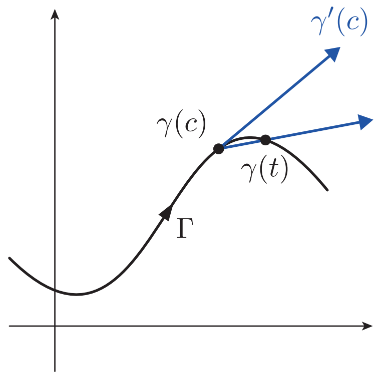


Figure 4.2 Tangent vector to a path

Tangent vectors to paths

Let Γ be a path with parametrisation $\gamma\colon I \longrightarrow \mathbb{C}$, and suppose that $c \in I$. If γ is differentiable at c and if $\gamma'(c) \neq 0$, then $\gamma'(c)$ can be interpreted geometrically as a *tangent vector* to the path Γ at the point $\gamma(c)$.

It is often useful to think of the tangent vector that represents $\gamma'(c)$ as the velocity vector of a particle moving along the path. If $\gamma(t)$ represents the position of the particle on the path Γ at time t, then $\gamma(t) - \gamma(c)$ is the net displacement of the particle that occurs between times c and t. During this time, the velocity of the particle is approximately equal to the difference quotient

$$\frac{\gamma(t) - \gamma(c)}{t - c}.$$

The limit $\gamma'(c)$ as t tends to c is, therefore, the *velocity* of the particle at time c, expressed as a complex number (rather than a vector). With this interpretation, the direction of the tangent vector associated with $\gamma'(c)$ indicates the direction in which the particle is moving, and the length of the vector $|\gamma'(c)|$ indicates its speed.

Next we describe how to find the derivative of a parametrisation $\gamma\colon I \longrightarrow \mathbb{C}$ (when this derivative exists). In many cases, the most convenient method is to notice that the parametrisation γ is the restriction to I of a complex function whose domain is a region that contains I.

For example, the parametrisation

$$\gamma(t) = (1 + it)^3 \quad (t \in [0, 2])$$

is the restriction to $I = [0, 2]$ of the function

$$f(z) = (1 + iz)^3,$$

whose domain is $\mathbb{C}$. Since

$$f'(z) = 3i(1 + iz)^2,$$

it follows by the Restriction Rule that

$$\gamma'(t) = 3i(1 + it)^2 \quad (t \in [0, 2]).$$

In practice, in such cases, we can treat t as if it were the complex variable z and invoke the Restriction Rule. (There is no need to set up the function f; we just 'differentiate $\gamma(t)$ with respect to t'.)

Exercise 4.2

For each of the following parametrisations, find the derivative and calculate $\gamma'(0)$. In each case sketch the parametrisation and mark the tangent vector $\gamma'(0)$ by an arrow emanating from the point $\gamma(0)$.

(a) $\gamma(t) = t + i(3 - t) \quad (t \in [-1, 2])$

(b) $\gamma(t) = \cos t + 2i \sin t \quad (t \in [0, 2\pi])$

(c) $\gamma(t) = t^2 + 2it \quad (t \in \mathbb{R})$

(d) $\gamma(t) = 2\cosh t + 3i \sinh t \quad (t \in \mathbb{R})$

(e) $\gamma(t) = e^{it} \quad (t \in [0, 2\pi])$

The next result gives a useful method for spotting points at which a parametrisation is *not* differentiable.

Theorem 4.1

Let ϕ and ψ be real functions, both with domain an interval I. Then the parametrisation

$$\gamma(t) = \phi(t) + i\,\psi(t) \quad (t \in I)$$

is differentiable at a point $c \in I$ if and only if both ϕ and ψ are differentiable at c. If ϕ and ψ are differentiable at c, then

$$\gamma'(c) = \phi'(c) + i\,\psi'(c).$$

Proof If the real functions ϕ and ψ are both differentiable at c, then, by the Sum and Multiple Rules for differentiation, γ is also differentiable at c and

$$\gamma'(c) = \phi'(c) + i\psi'(c).$$

On the other hand, if γ is differentiable at c, then, for real values t,

$$\begin{aligned}
\phi'(c) &= \lim_{t\to c} \frac{\phi(t)-\phi(c)}{t-c} \\
&= \lim_{t\to c} \frac{\operatorname{Re}(\gamma(t)) - \operatorname{Re}(\gamma(c))}{t-c} \\
&= \lim_{t\to c} \operatorname{Re}\left(\frac{\gamma(t)-\gamma(c)}{t-c}\right) \\
&= \operatorname{Re}\left(\lim_{t\to c} \frac{\gamma(t)-\gamma(c)}{t-c}\right) \\
&= \operatorname{Re}(\gamma'(c)).
\end{aligned}$$

In the second-to-last line, we used the observation that if (z_n) is a convergent sequence with limit $\alpha = \lim_{n\to\infty} z_n$, then

$$\lim_{n\to\infty} \operatorname{Re} z_n = \operatorname{Re}\alpha = \operatorname{Re}\left(\lim_{n\to\infty} z_n\right),$$

by Theorem 3.1 of Unit A3, because the real part function is continuous on $\mathbb{C}$.

Using a similar argument, we can see that $\psi'(c) = \operatorname{Im}(\gamma'(c))$.
Thus ϕ and ψ are differentiable at c, as required. ■

Remark

This theorem does not apply to functions with domains that are not contained in $\mathbb{R}$. For example, it is not possible to use the equation

$$z = \operatorname{Re} z + i \operatorname{Im} z$$

to deduce that Re and Im are differentiable. They are not!

Theorem 4.1 states that the differentiability of γ is *equivalent* to the differentiability of its real and imaginary part functions. It follows that a parametrisation like

$$\gamma(t) = t + i|t| \quad (t \in \mathbb{R})$$

is not differentiable at 0, since the real modulus function is not differentiable at 0. However, γ is differentiable on $\mathbb{R} - \{0\}$, since its real and imaginary part functions are both differentiable there.

Exercise 4.3

Determine the points at which the following parametrisation is differentiable:

$$\gamma(t) = t + i\sqrt{t} \quad (t \in [0,\infty)).$$

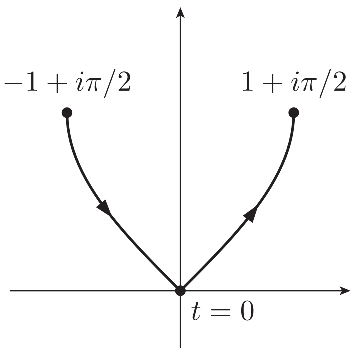


Figure 4.3 Path that is not differentiable at the origin

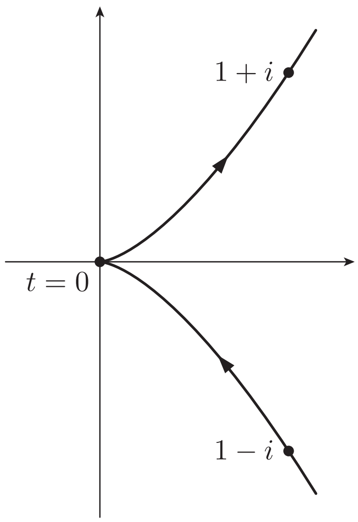


Figure 4.4 Path with derivative zero at the origin

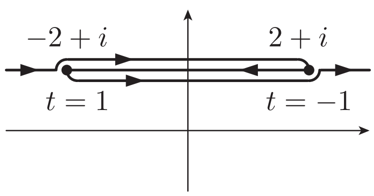


Figure 4.5 Path with derivative zero at $-2 + i$ and $2 + i$

Looking at the path with parametrisation

$$\gamma(t) = \sin t + i|t| \quad (t \in [-\pi/2, \pi/2]),$$

which you sketched in Exercise 4.1(b), and which is reproduced in Figure 4.3, you can see that the kink in the path at $t = 0$ coincides with the point at which γ fails to be differentiable. It is sometimes convenient to avoid such kinks by confining our attention to paths with a parametrisation whose derivative exists and varies continuously along the path. Actually, we need slightly more than this because kinks can also occur when the derivative of a parametrisation is zero. For example, Figure 4.4 shows the path with the parametrisation

$$\gamma(t) = t^2 + it^3 \quad (t \in \mathbb{R})$$

(sketched in Example 2.1 of Unit A2). Although this parametrisation is differentiable on its domain, its derivative is zero at the kink. In terms of the particle analogy, a zero derivative gives rise to a point on the path where the particle stops instantaneously. At this point, it is able to change direction abruptly, without upsetting the continuity of the velocity.

Another example of where a zero derivative can lead to difficulties is the path with the parametrisation

$$\gamma(t) = \left(t^3 - 3t\right) + i \quad (t \in \mathbb{R}).$$

Figure 4.5 shows sections of this path between $-2 + i$ and $2 + i$ just above and below the line $\operatorname{Im} z = 1$ to distinguish them (even though really the path lies wholly on the line $\operatorname{Im} z = 1$). The derivative of γ is zero when t is equal to -1 or 1, so the particle is able to 'reverse' at these points and retrace points already covered.

When we wish to avoid the kinds of behaviour exhibited in Figures 4.3–4.5, we confine our attention to paths that are *smooth* in the following sense.

Definitions

A parametrisation $\gamma\colon I \longrightarrow \mathbb{C}$ is **smooth** if

- γ is differentiable on I
- γ' is continuous on I
- γ' is non-zero on I.

A path is **smooth** if it has a smooth parametrisation.

Exercise 4.4

Decide which of the following parametrisations are smooth.

(a) $\gamma(t) = t + i(1 - \cos t) \quad (t \in [0, 2\pi])$

(b) $\gamma(t) = t^2 - 2t + i\pi \quad (t \in [0, 2])$

(c) $\gamma(t) = |t| + it \quad (t \in [-1, 1])$

In the next unit we will define *integration along a smooth path*. Here we use smooth paths to pursue the geometric interpretation of derivatives that we began in Subsection 1.5.

4.2 A geometric interpretation of derivatives (revisited)

In Subsection 1.5 we interpreted the derivative of an analytic function by saying that, to a close approximation, a small disc centred at α is mapped to a small disc centred at $f(\alpha)$ (provided that $f'(\alpha) \neq 0$). In the process, the disc is rotated through the angle $\operatorname{Arg} f'(\alpha)$, and scaled by the factor $|f'(\alpha)|$ (see Figure 4.6).

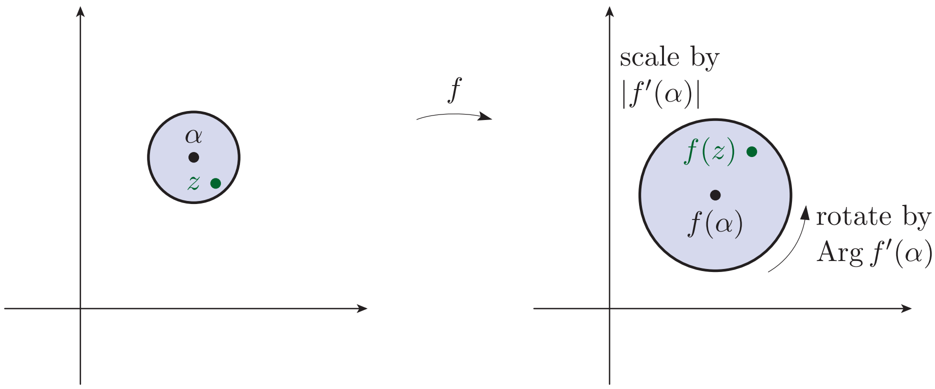


Figure 4.6 The approximate image of a small disc under an analytic function

This interpretation of $f'(\alpha)$ as a rotation and a scaling is all very well, but it is only an approximation. Fortunately, we can make the interpretation precise by examining the effect that f has on the tangent vectors to smooth paths through α.

Let f be a function that is analytic on a region $\mathcal{R}$, and let $\gamma\colon I \longrightarrow \mathbb{C}$ be a parametrisation of a smooth path Γ contained in $\mathcal{R}$. Then $f \circ \gamma\colon I \longrightarrow \mathbb{C}$ is a parametrisation of the image path $f(\Gamma)$ (see Figure 4.7); it is continuous because it is the composite of two continuous functions.

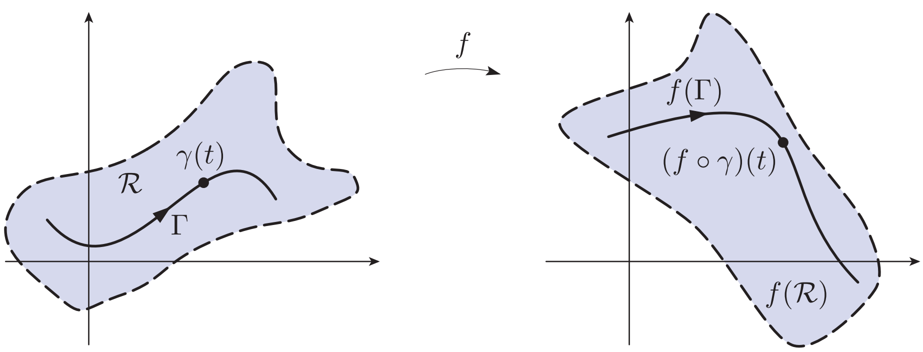


Figure 4.7 The image of the path Γ under the function f

Now consider a point $\alpha = \gamma(c)$ on the path Γ. This is mapped by f to the point $f(\alpha) = (f \circ \gamma)(c)$ on the path $f(\Gamma)$. Since the path Γ is smooth, a tangent vector to Γ at α is given by the derivative $\gamma'(c)$. Also, if $f'(\alpha) \neq 0$, then a tangent vector to $f(\Gamma)$ at $f(\alpha)$ is given by

$$(f \circ \gamma)'(c) = f'(\gamma(c))\,\gamma'(c) = f'(\alpha)\,\gamma'(c),$$

by the Chain Rule, as shown in Figure 4.8.

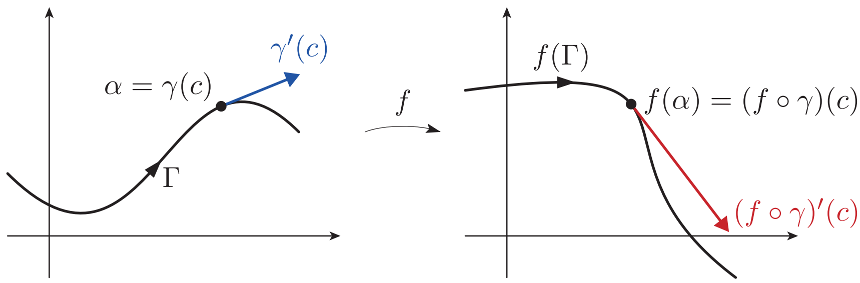


Figure 4.8 Image of a tangent vector to a path under an analytic function

We have thus shown that the two tangent vectors are related by the complex scale factor $f'(\alpha)$, in the following way.

Images of tangent vectors

Let f be a function that is analytic on a region $\mathcal{R}$, and suppose that $f'(\alpha) \neq 0$ for some $\alpha \in \mathcal{R}$.

If Γ is a smooth path in $\mathcal{R}$ that passes through α, then the tangent vector to the image path $f(\Gamma)$ at $f(\alpha)$ can be obtained from the tangent vector to Γ at α by a rotation through the angle $\operatorname{Arg} f'(\alpha)$ and a scaling by the factor $|f'(\alpha)|$.

As usual, the rotation is anticlockwise if $\operatorname{Arg} f'(\alpha)$ is positive, and clockwise if it is negative.

This is the geometric interpretation of derivative that we have been seeking. It is no longer an approximate result about the effect that f has on small discs, but a precise statement about the effect that f has on tangent vectors to paths. It is summarised in Figure 4.9.

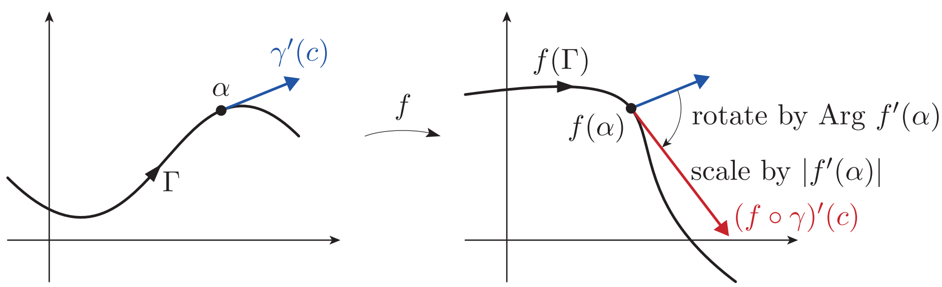


Figure 4.9 A tangent vector is rotated and scaled by the function f

We remark that if $f'(\alpha) = 0$, then the parametrisation $f \circ \gamma$ of the image path $f(\Gamma)$ is *not* smooth. This is because

$$(f \circ \gamma)'(c) = f'(\gamma(c))\,\gamma'(c) = f'(\alpha)\,\gamma'(c) = 0,$$

so it is not true that $(f \circ \gamma)'$ is non-zero on I.

4.3 Conformal functions

Suppose that Γ_1 and Γ_2 are two smooth paths with parametrisations $\gamma_1 : I_1 \longrightarrow \mathbb{C}$ and $\gamma_2 : I_2 \longrightarrow \mathbb{C}$ that intersect at the point $\alpha = \gamma_1(t_1) = \gamma_2(t_2)$. Then $\gamma_1'(t_1)$ and $\gamma_2'(t_2)$ can be interpreted as being tangent vectors to Γ_1 and Γ_2 at α. The angle $\theta \in (-\pi, \pi]$ from the tangent of one path to the other at α serves as a measure of the angle at which the paths themselves intersect (see Figure 4.10(a)).

To determine this angle, observe that

$$\gamma_2'(t_2) = \left(\frac{\gamma_2'(t_2)}{\gamma_1'(t_1)}\right)\gamma_1'(t_1).$$

From this we see that the sum of $\operatorname{Arg}(\gamma_2'(t_2)/\gamma_1'(t_1))$ and any argument of $\gamma_1'(t_1)$ is an argument of $\gamma_2'(t_2)$, so $\operatorname{Arg}(\gamma_2'(t_2)/\gamma_1'(t_1))$ is the angle from the vector $\gamma_1'(t_1)$ to the vector $\gamma_2'(t_2)$ that lies in the interval $(-\pi, \pi]$.

Definition

The **angle from Γ_1 to Γ_2 at α** is

$$\theta = \operatorname{Arg}\left(\frac{\gamma_2'(t_2)}{\gamma_1'(t_1)}\right).$$

In fact, the formula above specifies the angle from Γ_1 to Γ_2 at α *that lies in the interval* $(-\pi, \pi]$, because the image set of Arg is $(-\pi, \pi]$. We refer to this particular angle $\theta \in (-\pi, \pi]$ as 'the angle from Γ_1 to Γ_2 at α', and ignore other, equivalent angles $\theta + 2n\pi$, where n is a non-zero integer.

If we interchange Γ_1 and Γ_2, and instead consider the angle from Γ_2 to Γ_1 at α, then we find that (unless $\theta = \pi$) this new angle is $-\theta$, instead of θ, as illustrated in Figure 4.10(b). (If $\theta = \pi$, then the new angle is π too.)

Notice also that we obtain a different angle from Γ_1 to Γ_2 if we reverse the direction of one or both of the paths.

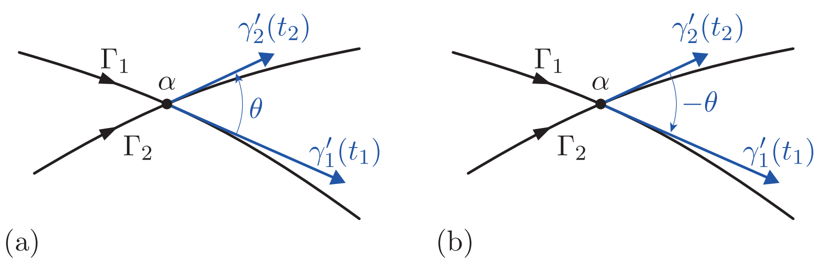


Figure 4.10 (a) The angle from Γ_1 to Γ_2 at α is θ (b) The angle from Γ_2 to Γ_1 at α is $-\theta$

Example 4.1

Let Γ_1 and Γ_2 be the paths with parametrisations

$$\gamma_1(t) = e^t + i(1+3t) \quad (t \in [-\tfrac{1}{2}, \tfrac{1}{2}]),$$
$$\gamma_2(t) = (t+2) + it^2 \quad (t \in [-\tfrac{3}{2}, \tfrac{1}{2}]),$$

respectively. Show that the two paths meet at the point $1+i$, and find the angle from Γ_1 to Γ_2 at this point of intersection.

Solution

The paths Γ_1 and Γ_2 meet at $1+i$ since

$$\gamma_1(0) = \gamma_2(-1) = 1+i.$$

Now

$$\gamma_1'(t) = e^t + 3i \quad \text{and} \quad \gamma_2'(t) = 1 + 2ti,$$

so

$$\gamma_1'(0) = 1+3i \quad \text{and} \quad \gamma_2'(-1) = 1 - 2i.$$

Hence the angle from Γ_1 to Γ_2 at $1+i$ is

$$\begin{aligned}\operatorname{Arg}\left(\frac{\gamma_2'(-1)}{\gamma_1'(0)}\right) &= \operatorname{Arg}\left(\frac{1-2i}{1+3i}\right)\\ &= \operatorname{Arg}\left(\tfrac{1}{10}(1-2i)(\overline{1+3i})\right)\\ &= \operatorname{Arg}((1-2i)(1-3i))\\ &= \operatorname{Arg}(-5-5i)\\ &= -3\pi/4.\end{aligned}$$

So the angle from Γ_1 to Γ_2 at $1+i$ is $-3\pi/4$, as shown in Figure 4.11.

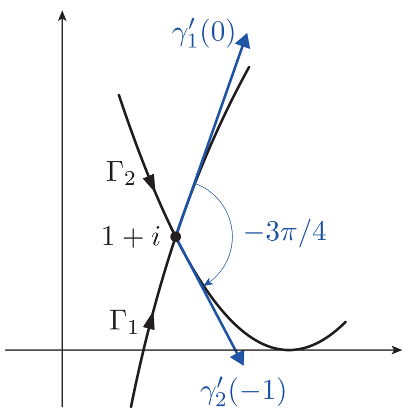


Figure 4.11 Paths Γ_1 and Γ_2 intersecting at $1+i$

Notice that the parameters for two intersecting paths are not necessarily equal at the point of intersection. For instance, in Example 4.1 we saw that $\gamma_1(0) = \gamma_2(-1) = 1+i$.

Now let us consider the effect of an analytic function f on the angle from one smooth path Γ_1 to another smooth path Γ_2 at a point α of intersection. Assuming that $f'(\alpha) \neq 0$, we see that the tangent vectors to the image paths $f(\Gamma_1)$ and $f(\Gamma_2)$ at $f(\alpha)$ are obtained from the tangent vectors to Γ_1 and Γ_2 at α by a rotation through the angle $\operatorname{Arg} f'(\alpha)$. Consequently, the angle from $f(\Gamma_1)$ to $f(\Gamma_2)$ at $f(\alpha)$ is equal to the angle from Γ_1 to Γ_2 at α (Figure 4.12).

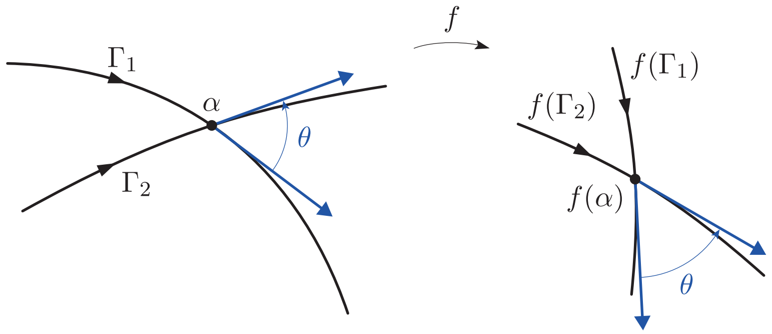


Figure 4.12 An analytic function preserving the angle between smooth paths

The angle-preserving property of analytic functions at points where their derivatives are non-zero is sufficiently important that it is given a special name.

Definitions

A function that is analytic at a point α is said to be **conformal at $\boldsymbol{\alpha}$** if the angle from any smooth path through α to any other smooth path through α is preserved by the function.

A function is **conformal on a set $\boldsymbol{S}$** if it is conformal at every point of S.

A function is **conformal** if it is conformal on its domain, in which case it is called a **conformal mapping**.

Suppose now that f is a function that is analytic at a point α. It is a consequence of the discussion above that if $f'(\alpha) \neq 0$, then f is conformal at α. In fact, we will see in Unit C2 that the converse to this statement is also true. Since the statement and its converse are useful, we present them together here in a theorem.

Theorem 4.2

Let f be a function that is analytic at a point α. Then f is conformal at α if and only if $f'(\alpha) \neq 0$.

We prove Theorem 4.2 later, in Subsection 3.3 of Unit C2.

Exercise 4.5

Give an example of two smooth paths through 0 such that the angle from one to the other at 0 is not preserved under the function $f(z) = z^2$. Why does this not contradict Theorem 4.2?

A striking illustration of Theorem 4.2 occurs when Cartesian and polar grids are used to investigate the behaviour of complex functions. Several functions were investigated in this way in Section 3 of Unit A2. The results obtained for the functions $f(z) = z^2$ and $f(z) = 1/z$ are reproduced in Figures 4.13 and 4.14.

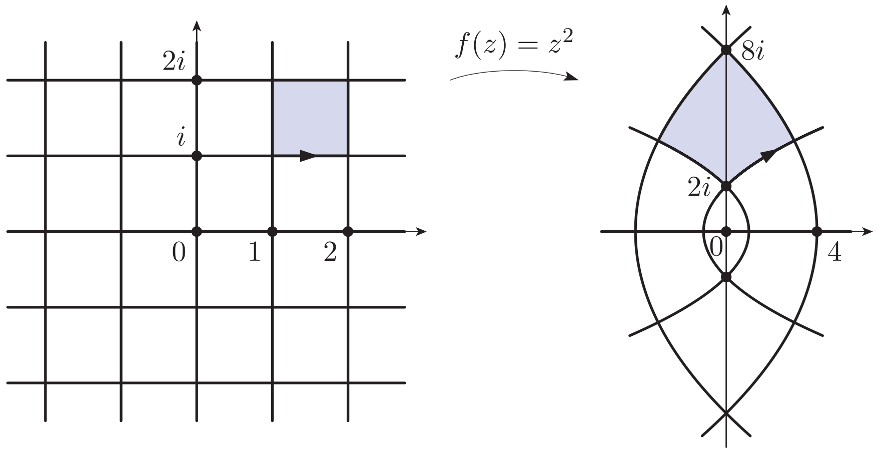


Figure 4.13 Image of a Cartesian grid under $f(z) = z^2$

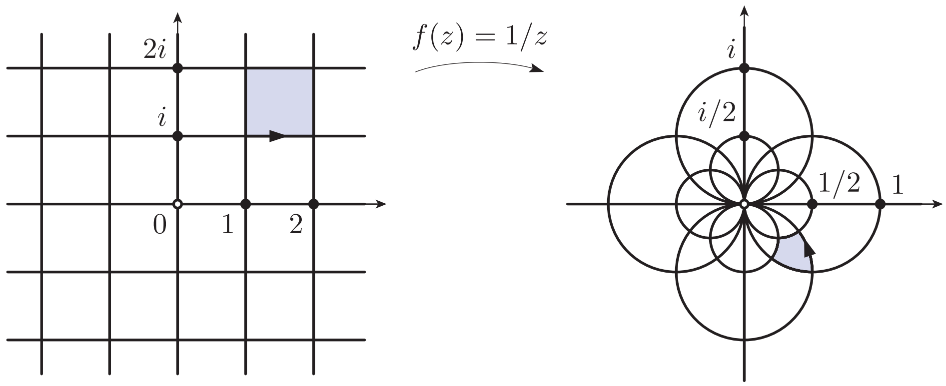


Figure 4.14 Image of a Cartesian grid under $f(z) = 1/z$

In both cases, the image of the Cartesian grid consists of paths that intersect at right angles (except at 0). Theorem 4.2 shows that this remarkable property of angle preservation holds for *all* analytic functions, provided that the derivative of the function is non-zero on the region under consideration.

Similarly, polar grids consist of paths that intersect at right angles, so the image of a polar grid under an analytic function with non-zero derivative on a region containing the grid is another collection of paths that intersect at right angles.

Motivated by these observations, we make the following definition.

Definition

Smooth paths that meet at right angles are said to be **orthogonal**. An **orthogonal grid** is a grid made up of orthogonal smooth paths.

Theorem 4.2 shows that an analytic function f maps an orthogonal grid, over any region where f' is non-zero, to an orthogonal grid.

We will return to the conformal properties of analytic functions later in the module.

Further exercises

Exercise 4.6

Find the derivatives of each of the following parametrisations.

(a) $\gamma(t) = \exp(t^2 + it) \quad (t \in [-1, 1])$

(b) $\gamma(t) = t^2 + i\cos t \quad (t \in [-1, 1])$

(c) $\gamma(t) = 5\cos t + 7i\sin t \quad (t \in [-\pi, \pi])$

Exercise 4.7

Which of the parametrisations in Exercise 4.6 are smooth?

Exercise 4.8

(a) Let Γ_1 and Γ_2 be the paths with parametrisations

$$\gamma_1(t) = 2\sin t + i(t+1) \quad (t \in [-1, 1]),$$
$$\gamma_2(t) = (1-t) + it^2 \quad (t \in [-1, 2]),$$

respectively. Show that the two paths meet at the point i, and find the angle from Γ_1 to Γ_2 at this point of intersection.

(b) Now let $f(z) = \sin^2 z$. Determine the angle from $f(\Gamma_1)$ to $f(\Gamma_2)$ at $f(i)$.

Solutions to exercises

Solution to Exercise 1.1

(a) $f(z) = 1$ is defined on the whole of $\mathbb{C}$, so let $\alpha \in \mathbb{C}$. Then

$$\begin{aligned} f'(\alpha) &= \lim_{z\to\alpha} \frac{f(z)-f(\alpha)}{z-\alpha} \\ &= \lim_{z\to\alpha} \frac{1-1}{z-\alpha} \\ &= \lim_{z\to\alpha} \frac{0}{z-\alpha} = 0. \end{aligned}$$

Since α is an arbitrary complex number, f is differentiable on the whole of $\mathbb{C}$, and its derivative is the zero function

$$f'(z) = 0 \quad (z \in \mathbb{C}).$$

(b) $f(z) = z$ is defined on the whole of $\mathbb{C}$, so let $\alpha \in \mathbb{C}$. Then

$$\begin{aligned} f'(\alpha) &= \lim_{z\to\alpha} \frac{f(z)-f(\alpha)}{z-\alpha} \\ &= \lim_{z\to\alpha} \frac{z-\alpha}{z-\alpha} \\ &= \lim_{z\to\alpha} 1 = 1. \end{aligned}$$

Since α is an arbitrary complex number, f is differentiable on the whole of $\mathbb{C}$, and its derivative is the constant function

$$f'(z) = 1 \quad (z \in \mathbb{C}).$$

Solution to Exercise 1.2

The domain of $f(z) = 1/z$ is the region $\mathbb{C} - \{0\}$. Since $f'(\alpha)$ cannot exist unless f is defined at α, we confine our attention to $\alpha \neq 0$. Then

$$\begin{aligned} f'(\alpha) &= \lim_{z\to\alpha} \frac{f(z)-f(\alpha)}{z-\alpha} \\ &= \lim_{z\to\alpha} \frac{(1/z)-(1/\alpha)}{z-\alpha} \\ &= \lim_{z\to\alpha} \frac{\alpha - z}{z\alpha(z-\alpha)} \\ &= \lim_{z\to\alpha} \frac{-1}{z\alpha}. \end{aligned}$$

Now $z \longmapsto -1/(z\alpha)$ is a basic continuous function with domain $\mathbb{C} - \{0\}$, so we see (from Theorem 3.1 of Unit A3) that

$$f'(\alpha) = \lim_{z\to\alpha} \frac{-1}{z\alpha} = -\frac{1}{\alpha^2}.$$

Since α is an arbitrary non-zero complex number, the derivative of f is

$$f'(z) = -\frac{1}{z^2} \quad (z \neq 0).$$

The function f is not entire since its domain is not $\mathbb{C}$.

Solution to Exercise 1.3

(a) True.

(b) False. (The set must be a region.)

Solution to Exercise 1.4

(a) Let $F = f + g$. Then

$$\begin{aligned} &\lim_{z\to\alpha} \frac{F(z)-F(\alpha)}{z-\alpha} \\ &\quad= \lim_{z\to\alpha} \frac{(f(z)+g(z))-(f(\alpha)+g(\alpha))}{z-\alpha} \\ &\quad= \lim_{z\to\alpha} \frac{(f(z)-f(\alpha))+(g(z)-g(\alpha))}{z-\alpha} \\ &\quad= \lim_{z\to\alpha} \frac{f(z)-f(\alpha)}{z-\alpha} + \lim_{z\to\alpha} \frac{g(z)-g(\alpha)}{z-\alpha} \\ &\quad= f'(\alpha) + g'(\alpha). \end{aligned}$$

(b) Let $F = \lambda f$, for $\lambda \in \mathbb{C}$. Then

$$\begin{aligned} \lim_{z\to\alpha} \frac{F(z)-F(\alpha)}{z-\alpha} &= \lim_{z\to\alpha} \frac{\lambda f(z)-\lambda f(\alpha)}{z-\alpha} \\ &= \lambda \lim_{z\to\alpha} \frac{f(z)-f(\alpha)}{z-\alpha} \\ &= \lambda f'(\alpha). \end{aligned}$$

Solution to Exercise 1.5

(a) By the corollary on differentiating polynomial functions, we have

$$f'(z) = 4z^3 + 9z^2 - 2z + 4 \quad (z \in \mathbb{C}).$$

(b) By the Quotient Rule,

$$\begin{aligned} f'(z) &= \frac{(z^2+z+1)(2z-4)-(z^2-4z+2)(2z+1)}{(z^2+z+1)^2} \\ &= \frac{5z^2-2z-6}{(z^2+z+1)^2}. \end{aligned}$$

Now, $z^2 + z + 1 = 0$ if and only if $z = -\frac{1}{2}(1 \pm \sqrt{3}i)$, so the domain of f' is

$$\mathbb{C} - \{-\tfrac{1}{2}(1+\sqrt{3}i), -\tfrac{1}{2}(1-\sqrt{3}i)\}.$$

Solution to Exercise 1.6

The function Arg is discontinuous at each point of the negative real axis (see Exercise 2.4 of Unit A3). It follows that Log is discontinuous at each point of the negative real axis, and hence that there are no points on it at which Log is differentiable.

Solution to Exercise 1.7

Let $z_n = \alpha \exp(i/n)$, $n = 1, 2, \ldots$. Then (z_n) tends to α along the circumference of the circle, and

$$\lim_{n\to\infty} \frac{|z_n| - |\alpha|}{z_n - \alpha} = \lim_{n\to\infty} \frac{|\alpha| - |\alpha|}{z_n - \alpha} = 0.$$

Now let $z'_n = \alpha(1 + 1/n)$, $n = 1, 2, \ldots$. Then (z'_n) tends to α along the ray from 0 through α, and

$$\lim_{n\to\infty} \frac{|z'_n| - |\alpha|}{z'_n - \alpha} = \lim_{n\to\infty} \frac{|\alpha|(1 + 1/n) - |\alpha|}{\alpha(1 + 1/n) - \alpha} = \frac{|\alpha|}{\alpha}.$$

Since $|\alpha|/\alpha \neq 0$ for $\alpha \neq 0$, these two limits do not agree. It follows that $f(z) = |z|$ is not differentiable at $\alpha \neq 0$.

Solution to Exercise 1.8

Let α be an arbitrary complex number. Directions of paths parallel to the imaginary axis through α are reversed by f, while directions of paths parallel to the real axis are not. This suggests looking at the sequences $z_n = \alpha + 1/n$ and $z'_n = \alpha + i/n$, $n = 1, 2, \ldots$.

First let $z_n = \alpha + 1/n$; then

$$\begin{aligned}\lim_{n\to\infty} \frac{\overline{z_n} - \overline{\alpha}}{z_n - \alpha} &= \lim_{n\to\infty} \frac{\overline{(\alpha + 1/n)} - \overline{\alpha}}{(\alpha + 1/n) - \alpha} \\ &= \lim_{n\to\infty} \frac{1/n}{1/n} = 1.\end{aligned}$$

Now let $z'_n = \alpha + i/n$; then

$$\begin{aligned}\lim_{n\to\infty} \frac{\overline{z'_n} - \overline{\alpha}}{z'_n - \alpha} &= \lim_{n\to\infty} \frac{\overline{(\alpha + i/n)} - \overline{\alpha}}{(\alpha + i/n) - \alpha} \\ &= \lim_{n\to\infty} \frac{-i/n}{i/n} = -1.\end{aligned}$$

Since these two limits do not agree, and since α is arbitrary, it follows that there are no points of $\mathbb{C}$ at which $f(z) = \overline{z}$ is differentiable.

Solution to Exercise 1.9

(a) The fact that $\operatorname{Re} z$ is constant along the imaginary axis, but variable parallel to the real axis, suggests that Re is not differentiable at i (or anywhere else, for that matter). It also suggests looking at the sequences $z_n = i + i/n$ and $z'_n = i + 1/n$, $n = 1, 2, \ldots$.

First let $z_n = i + i/n$; then

$$\begin{aligned}\lim_{n\to\infty} \frac{\operatorname{Re} z_n - \operatorname{Re} i}{z_n - i} &= \lim_{n\to\infty} \frac{\operatorname{Re}(i + i/n) - \operatorname{Re} i}{(i + i/n) - i} \\ &= \lim_{n\to\infty} \frac{0}{i/n} = 0.\end{aligned}$$

Now let $z'_n = i + 1/n$; then

$$\begin{aligned}\lim_{n\to\infty} \frac{\operatorname{Re} z'_n - \operatorname{Re} i}{z'_n - i} &= \lim_{n\to\infty} \frac{\operatorname{Re}(i + 1/n) - \operatorname{Re} i}{(i + 1/n) - i} \\ &= \lim_{n\to\infty} \frac{1/n}{1/n} = 1.\end{aligned}$$

Since these two limits do not agree, it follows that Re is not differentiable at i.

(b) f is a polynomial function, so $f'(z) = 4z + 3$ for all $z \in \mathbb{C}$. Thus $f'(i) = 3 + 4i$.

(c) f is not differentiable at i, since it is not continuous at i.

Solution to Exercise 1.10

To a close approximation, a small disc centred at i is mapped by f to a small disc centred at

$$f(i) = \frac{4i + 3}{2i^2 + 1} = -3 - 4i.$$

In the process the disc is scaled by the factor $|f'(i)|$ and rotated through the angle $\operatorname{Arg} f'(i)$.

By the Quotient Rule,

$$\begin{aligned}f'(z) &= \frac{4(2z^2 + 1) - 4z(4z + 3)}{(2z^2 + 1)^2} \\ &= \frac{-8z^2 - 12z + 4}{(2z^2 + 1)^2}.\end{aligned}$$

So

$$f'(i) = \frac{-8i^2 - 12i + 4}{(2i^2 + 1)^2} = 12 - 12i.$$

This has modulus $12\sqrt{2}$ and principal argument $-\pi/4$.

So f scales the disc by the factor $12\sqrt{2}$ and rotates it clockwise through the angle $\pi/4$.

Solution to Exercise 1.11

The function $f(z) = 2z^2 + 5$ is defined on the whole of $\mathbb{C}$. Let $\alpha \in \mathbb{C}$. Then

$$\begin{aligned} f'(\alpha) &= \lim_{z \to \alpha} \frac{f(z) - f(\alpha)}{z - \alpha} \\ &= \lim_{z \to \alpha} \frac{(2z^2 + 5) - (2\alpha^2 + 5)}{z - \alpha} \\ &= \lim_{z \to \alpha} \frac{2(z^2 - \alpha^2)}{z - \alpha} \\ &= \lim_{z \to \alpha} 2(z + \alpha) \\ &= 4\alpha. \end{aligned}$$

Since α is an arbitrary complex number, f is differentiable on the whole of $\mathbb{C}$, and the derivative is the function

$$f'(z) = 4z \quad (z \in \mathbb{C}).$$

Solution to Exercise 1.12

Let $F = f/g$. Then

$$\begin{aligned} &\frac{F(z) - F(\alpha)}{z - \alpha} \\ &= \frac{f(z)/g(z) - f(\alpha)/g(\alpha)}{z - \alpha} \\ &= \frac{f(z)g(\alpha) - f(\alpha)g(z)}{(z - \alpha)g(z)g(\alpha)} \\ &= \frac{g(\alpha)(f(z) - f(\alpha)) - f(\alpha)(g(z) - g(\alpha))}{(z - \alpha)g(z)g(\alpha)} \\ &= \frac{g(\alpha)\left(\dfrac{f(z) - f(\alpha)}{z - \alpha}\right) - f(\alpha)\left(\dfrac{g(z) - g(\alpha)}{z - \alpha}\right)}{g(z)g(\alpha)}. \end{aligned}$$

Using the Combination Rules for limits of functions, the continuity of g, and the fact that $g(\alpha) \neq 0$, we can take limits to obtain

$$F'(\alpha) = \frac{g(\alpha)f'(\alpha) - f(\alpha)g'(\alpha)}{(g(\alpha))^2}.$$

Solution to Exercise 1.13

(a) By the Combination Rules,

$$\begin{aligned} f'(z) &= \frac{(3z + 1)(2z + 2) - 3(z^2 + 2z + 1)}{(3z + 1)^2} \\ &= \frac{3z^2 + 2z - 1}{(3z + 1)^2}. \end{aligned}$$

The domain of f' is $\mathbb{C} - \{-1/3\}$.

(b) By the Combination Rules,

$$\begin{aligned} f'(z) &= \frac{(z^2 - z - 6)(3z^2) - (z^3 + 1)(2z - 1)}{(z^2 - z - 6)^2} \\ &= \frac{z^4 - 2z^3 - 18z^2 - 2z + 1}{(z^2 - z - 6)^2}. \end{aligned}$$

Since $z^2 - z - 6 = (z + 2)(z - 3)$, the domain of f' is $\mathbb{C} - \{-2, 3\}$.

(c) By the Reciprocal Rule,

$$f'(z) = \frac{-(2z + 2)}{(z^2 + 2z + 2)^2}.$$

The roots of $z^2 + 2z + 2$ are $\dfrac{-2 \pm \sqrt{-4}}{2} = -1 \pm i$.
The domain of f' is therefore $\mathbb{C} - \{-1 + i, -1 - i\}$.

(d) By the Sum Rule and the rule for differentiating integer powers,

$$f'(z) = 2z + 5 - \frac{1}{z^2} - \frac{2}{z^3}.$$

The domain of f' is $\mathbb{C} - \{0\}$.

Solution to Exercise 1.14

Consider an arbitrary complex number $\alpha = a + ib$, where $a, b \in \mathbb{R}$. Let $z_n = \alpha + 1/n$, $n = 1, 2, \ldots$. Then $z_n \to \alpha$, and

$$\begin{aligned} \lim_{n \to \infty} \frac{\operatorname{Im} z_n - \operatorname{Im} \alpha}{z_n - \alpha} &= \lim_{n \to \infty} \frac{b - b}{1/n} \\ &= \lim_{n \to \infty} \frac{0}{1/n} = 0. \end{aligned}$$

Now let $z'_n = \alpha + i/n$, $n = 1, 2, \ldots$. Then $z'_n \to \alpha$, and

$$\begin{aligned} \lim_{n \to \infty} \frac{\operatorname{Im} z'_n - \operatorname{Im} \alpha}{z'_n - \alpha} &= \lim_{n \to \infty} \frac{(b + 1/n) - b}{i/n} \\ &= \lim_{n \to \infty} \frac{1/n}{i/n} = -i. \end{aligned}$$

Since the two limits do not agree, it follows that Im fails to be differentiable at each point of $\mathbb{C}$.

Solution to Exercise 1.15

To a close approximation, a small disc centred at 2 is mapped by f to small disc centred at $f(2) = -4$. In the process, the disc is scaled by the factor $|f'(2)|$ and rotated through the angle $\operatorname{Arg} f'(2)$.

By the Quotient Rule,

$$f'(z) = \frac{3z^2(z-6)-(z^3+8)}{(z-6)^2} = \frac{2z^3-18z^2-8}{(z-6)^2}.$$

So f scales the disc by the factor 4 and rotates it anticlockwise through the angle π.

Solution to Exercise 2.1

(a) Differentiating $v(x,y) = 3x^2y - y^3$ with respect to x while keeping y fixed, we obtain

$$\frac{\partial v}{\partial x}(x,y) = 6xy.$$

Differentiating v with respect to y while keeping x fixed, we obtain

$$\frac{\partial v}{\partial y}(x,y) = 3x^2 - 3y^2.$$

(b) So, at $(x,y) = (2,1)$ the partial derivatives have the values

$$\frac{\partial v}{\partial x}(2,1) = 12 \quad \text{and} \quad \frac{\partial v}{\partial y}(2,1) = 9.$$

Solution to Exercise 2.2

(a) Writing f in the form

$$f(x+iy) = u(x,y) + iv(x,y),$$

we obtain

$$u(x,y) = e^x \quad \text{and} \quad v(x,y) = -e^y.$$

Hence

$$\frac{\partial u}{\partial x}(x,y) = e^x \quad \text{and} \quad \frac{\partial v}{\partial y}(x,y) = -e^y.$$

Since e^x is always positive, whereas $-e^y$ is always negative, the first of the Cauchy–Riemann equations fails to hold for each (x,y). It follows that f fails to be differentiable at all points of $\mathbb{C}$.

(b) Writing $f(z) = \overline{z} = x - iy$ in the form

$$f(x+iy) = u(x,y) + iv(x,y),$$

we obtain

$$u(x,y) = x \quad \text{and} \quad v(x,y) = -y.$$

Hence

$$\frac{\partial u}{\partial x}(x,y) = 1 \quad \text{and} \quad \frac{\partial v}{\partial y}(x,y) = -1.$$

It follows that the first of the Cauchy–Riemann equations fails to hold for each (x,y), so f fails to be differentiable at all points of $\mathbb{C}$.

Solution to Exercise 2.3

(a) From the trigonometric identities in Unit A2,

$$\begin{aligned}\sin(x+iy) &= \sin x \cos iy + \cos x \sin iy \\ &= \sin x \cosh y + i \cos x \sinh y,\end{aligned}$$

so $f(x+iy) = u(x,y) + iv(x,y)$, where

$$\begin{aligned}u(x,y) &= \sin x \cosh y \quad \text{and} \\ v(x,y) &= \cos x \sinh y.\end{aligned}$$

Hence

$$\begin{aligned}\frac{\partial u}{\partial x}(x,y) &= \cos x \cosh y, \\ \frac{\partial v}{\partial x}(x,y) &= -\sin x \sinh y, \\ \frac{\partial u}{\partial y}(x,y) &= \sin x \sinh y, \\ \frac{\partial v}{\partial y}(x,y) &= \cos x \cosh y.\end{aligned}$$

These partial derivatives are defined and continuous on the whole of $\mathbb{C}$. Furthermore,

$$\begin{aligned}\frac{\partial u}{\partial x}(x,y) &= \frac{\partial v}{\partial y}(x,y) \quad \text{and} \\ \frac{\partial v}{\partial x}(x,y) &= -\frac{\partial u}{\partial y}(x,y),\end{aligned}$$

so the Cauchy–Riemann equations are satisfied at every point of $\mathbb{C}$.

By the Cauchy–Riemann Converse Theorem, $f(z) = \sin z$ is entire, and

$$\begin{aligned}f'(x+iy) &= \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y) \\ &= \cos x \cosh y - i \sin x \sinh y \\ &= \cos x \cos iy - \sin x \sin iy \\ &= \cos(x+iy).\end{aligned}$$

Hence f' has domain $\mathbb{C}$ and $f'(z) = \cos z$. (You will see an easier way of finding this derivative in Section 3.)

(b) Here $f(x+iy) = |x+iy|^2 = x^2+y^2$, so

$$u(x,y) = x^2+y^2 \quad \text{and} \quad v(x,y) = 0.$$

Hence

$$\frac{\partial u}{\partial x}(x,y) = 2x, \quad \frac{\partial v}{\partial x}(x,y) = 0,$$
$$\frac{\partial u}{\partial y}(x,y) = 2y, \quad \frac{\partial v}{\partial y}(x,y) = 0.$$

The Cauchy–Riemann equations cannot be satisfied unless $2x = 0$ and $-2y = 0$, so f fails to be differentiable at all non-zero points of $\mathbb{C}$.

However, the Cauchy–Riemann equations *are* satisfied at $(0,0)$, and the partial derivatives are defined on $\mathbb{C}$ and continuous (at $(0,0)$), so by the Cauchy–Riemann Converse Theorem, f is differentiable at 0, and

$$\begin{aligned} f'(0) &= \frac{\partial u}{\partial x}(0,0) + i\frac{\partial v}{\partial x}(0,0) \\ &= 0 + i0 = 0. \end{aligned}$$

Thus f' has domain $\{0\}$ and $f'(0) = 0$.

(This is the example referred to in Subsection 1.1 of a function that is differentiable at a point, but not analytic at that point.)

Solution to Exercise 2.4

(a) Differentiating $u(x,y) = 3x + xy + 2x^2y^2$ with respect to x while keeping y fixed, we obtain

$$\frac{\partial u}{\partial x}(x,y) = 3 + y + 4xy^2.$$

Differentiating with respect to y while keeping x fixed, we obtain

$$\frac{\partial u}{\partial y}(x,y) = x + 4x^2y.$$

(b) Here $u(x,y) = x\cos y + \exp(xy)$, so

$$\frac{\partial u}{\partial x}(x,y) = \cos y + y\exp(xy) \quad \text{and}$$
$$\frac{\partial u}{\partial y}(x,y) = -x\sin y + x\exp(xy).$$

(c) Here $u(x,y) = (x+y)^3$, so

$$\frac{\partial u}{\partial x}(x,y) = 3(x+y)^2 \quad \text{and}$$
$$\frac{\partial u}{\partial y}(x,y) = 3(x+y)^2.$$

Solution to Exercise 2.5

(a) Here $u(x,y) = x^3y - y\cos y$, so

$$\frac{\partial u}{\partial x}(x,y) = 3x^2y \quad \text{and}$$
$$\frac{\partial u}{\partial y}(x,y) = x^3 - \cos y + y\sin y.$$

So, at $(x,y) = (1,0)$ the partial derivatives have the values

$$\frac{\partial u}{\partial x}(1,0) = 0 \quad \text{and} \quad \frac{\partial u}{\partial y}(1,0) = 0.$$

(b) Here $u(x,y) = ye^x - xy^3$, so

$$\frac{\partial u}{\partial x}(x,y) = ye^x - y^3 \quad \text{and}$$
$$\frac{\partial u}{\partial y}(x,y) = e^x - 3xy^2.$$

So, at $(x,y) = (1,0)$ the partial derivatives have the values

$$\frac{\partial u}{\partial x}(1,0) = 0 \quad \text{and} \quad \frac{\partial u}{\partial y}(1,0) = e.$$

Solution to Exercise 2.6

Since $u(x,y) = x^2 + 2xy$, it follows that

$$\frac{\partial u}{\partial x}(x,y) = 2x + 2y \quad \text{and} \quad \frac{\partial u}{\partial y}(x,y) = 2x.$$

The gradient of the graph at $(x,y) = (1,2)$ in the x-direction is

$$\frac{\partial u}{\partial x}(1,2) = 2 \times 1 + 2 \times 2 = 6.$$

The gradient of the graph at $(x,y) = (1,2)$ in the y-direction is

$$\frac{\partial u}{\partial y}(1,2) = 2 \times 1 = 2.$$

Solution to Exercise 2.7

Writing f in the form

$$f(x+iy) = u(x,y) + iv(x,y),$$

we obtain

$$u(x,y) = e^x \sin y \quad \text{and} \quad v(x,y) = e^x \cos y.$$

Hence

$$\frac{\partial u}{\partial x}(x,y) = e^x \sin y, \quad \frac{\partial v}{\partial x}(x,y) = e^x \cos y,$$
$$\frac{\partial u}{\partial y}(x,y) = e^x \cos y, \quad \frac{\partial v}{\partial y}(x,y) = -e^x \sin y.$$

If f is differentiable at $x+iy$, then the Cauchy–Riemann equations require that

$$e^x \sin y = -e^x \sin y \quad \text{and}$$
$$e^x \cos y = -e^x \cos y;$$

that is,

$$e^x \sin y = 0 \quad \text{and} \quad e^x \cos y = 0.$$

But e^x is never zero, so $\sin y = \cos y = 0$, which is impossible. It follows that there is no point of $\mathbb{C}$ at which f is differentiable.

Solution to Exercise 2.8

In this case,

$$u(x,y) = x^2 + x - y^2 \quad \text{and}$$
$$v(x,y) = 2xy + y,$$

so

$$\frac{\partial u}{\partial x}(x,y) = 2x+1, \quad \frac{\partial v}{\partial x}(x,y) = 2y,$$
$$\frac{\partial u}{\partial y}(x,y) = -2y, \quad \frac{\partial v}{\partial y}(x,y) = 2x+1.$$

These partial derivatives are defined and continuous on the whole of $\mathbb{C}$. Furthermore,

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) \quad \text{and}$$
$$\frac{\partial v}{\partial x}(x,y) = -\frac{\partial u}{\partial y}(x,y),$$

so the Cauchy–Riemann equations are satisfied at every point of $\mathbb{C}$.

By the Cauchy–Riemann Converse Theorem, f is entire, and

$$\begin{aligned} f'(x+iy) &= \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y) \\ &= (2x+1) + 2yi. \end{aligned}$$

(So $f'(z) = 2z+1$, and in fact $f(z) = z^2 + z$.)

Solution to Exercise 2.9

(a) Here

$$u(x,y) = x^2 + y^2 \quad \text{and} \quad v(x,y) = x^2 - y^2,$$

so

$$\frac{\partial u}{\partial x}(x,y) = 2x, \quad \frac{\partial v}{\partial x}(x,y) = 2x,$$
$$\frac{\partial u}{\partial y}(x,y) = 2y, \quad \frac{\partial v}{\partial y}(x,y) = -2y.$$

The Cauchy–Riemann equations are satisfied only if $x = -y$. So f cannot be differentiable at $x+iy$ unless $x = -y$. Since the partial derivatives above exist, and are continuous on $\mathbb{C}$ (and in particular when $x = -y$), it follows from the Cauchy–Riemann Converse Theorem that f is differentiable on the set $\{x+iy : x = -y\}$.

On this set,

$$\begin{aligned} f'(x+iy) &= \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y) \\ &= 2x + 2xi = 2x(1+i). \end{aligned}$$

(b) Here

$$u(x,y) = xy \quad \text{and} \quad v(x,y) = 0,$$

so

$$\frac{\partial u}{\partial x}(x,y) = y, \quad \frac{\partial v}{\partial x}(x,y) = 0,$$
$$\frac{\partial u}{\partial y}(x,y) = x, \quad \frac{\partial v}{\partial y}(x,y) = 0.$$

The Cauchy–Riemann equations are not satisfied unless $y = 0$ and $-x = 0$. So f is not differentiable except possibly at 0. Since the partial derivatives above exist, and are continuous at $(0,0)$, it follows from the Cauchy–Riemann Converse Theorem that f is differentiable at 0. Furthermore,

$$f'(0) = \frac{\partial u}{\partial x}(0,0) + i\frac{\partial v}{\partial x}(0,0) = 0.$$

Solution to Exercise 3.1

(a) Here $k = g \circ f$, where

$$g(z) = z^{900} \quad \text{and} \quad f(z) = z+5.$$

Both f and g are entire, and

$$g'(z) = 900z^{899}, \quad f'(z) = 1.$$

So, by the Chain Rule, k is also entire and

$$\begin{aligned} k'(z) &= g'(f(z))f'(z) \\ &= 900(z+5)^{899} \times 1 = 900(z+5)^{899}. \end{aligned}$$

(b) Here $k = g \circ f$, where

$$g(z) = \exp z \quad \text{and} \quad f(z) = z^2 + 4.$$

Both f and g are entire, and

$$g'(z) = \exp z, \quad f'(z) = 2z.$$

So, by the Chain Rule, k is also entire and

$$k'(z) = g'(f(z))f'(z) = 2z\exp(z^2+4).$$

(c) Here $k = g \circ f$, where

$$g(z) = \exp z \quad \text{and} \quad f(z) = \alpha z.$$

Both f and g entire, and

$$g'(z) = \exp z, \quad f'(z) = \alpha.$$

So, by the Chain Rule, k is also entire and

$$\begin{aligned} k'(z) &= g'(f(z))f'(z) \\ &= \exp(\alpha z) \times \alpha = \alpha e^{\alpha z}. \end{aligned}$$

Solution to Exercise 3.2

The functions $z \longmapsto e^z$ and $z \longmapsto e^{-z}$ are entire with derivatives $z \longmapsto e^z$ and $z \longmapsto -e^{-z}$, respectively. It follows from the Combination Rules for differentiation that:

(a) $f(z) = \sinh z = (e^z - e^{-z})/2$ is entire, and

$$f'(z) = \frac{e^z - (-e^{-z})}{2} = \frac{e^z + e^{-z}}{2} = \cosh z$$

(b) $f(z) = \cosh z = (e^z + e^{-z})/2$ is entire, and

$$f'(z) = \frac{e^z - e^{-z}}{2} = \sinh z$$

(c) $f(z) = \tanh z = \sinh z / \cosh z$ is analytic on its domain, and

$$\begin{aligned} f'(z) &= \frac{\cosh z \times \cosh z - \sinh z \times \sinh z}{\cosh^2 z} \\ &= \frac{1}{\cosh^2 z} = \operatorname{sech}^2 z. \end{aligned}$$

Solution to Exercise 3.3

(a) (i) Here $k = h \circ g \circ f$, where

$$h(z) = z^2, \quad g(z) = z^2 + 3, \quad f(z) = \sin z.$$

All of f, g and h are entire, and

$$h'(z) = 2z, \quad g'(z) = 2z, \quad f'(z) = \cos z.$$

So, by the extended form of the Chain Rule, k is entire and

$$\begin{aligned} k'(z) &= h'(g(f(z))) \times g'(f(z)) \times f'(z) \\ &= 2(\sin^2 z + 3) \times 2 \sin z \times \cos z \\ &= 4 \sin z \cos z (\sin^2 z + 3). \end{aligned}$$

(ii) Here $k = h \circ g \circ f$, where

$$h(z) = \sin z, \quad g(z) = \exp z, \quad f(z) = \cos z - z.$$

Now f, g and h are entire, and

$$\begin{aligned} h'(z) &= \cos z, \\ g'(z) &= \exp z, \\ f'(z) &= -\sin z - 1. \end{aligned}$$

So, by the extended form of the Chain Rule, k is entire and

$$\begin{aligned} k'(z) &= h'(g(f(z))) \times g'(f(z)) \times f'(z) \\ &= -\cos(\exp(\cos z - z)) \times \exp(\cos z - z) \\ &\quad \times (\sin z + 1). \end{aligned}$$

(b) (i) k is differentiable on $\mathbb{C}$ and

$$k'(z) = \cos(\cosh z) \times \sinh z.$$

(ii) k is differentiable on $\mathbb{C}$ and

$$\begin{aligned} k'(z) &= -\sin((1+z)^{20}) \times 20(1+z)^{19} \times 1 \\ &= -20(1+z)^{19} \sin((1+z)^{20}). \end{aligned}$$

(iii) k is differentiable on $\mathbb{C}$ and

$$k'(z) = \exp(\exp(\sin z)) \times \exp(\sin z) \times \cos z.$$

Solution to Exercise 3.4

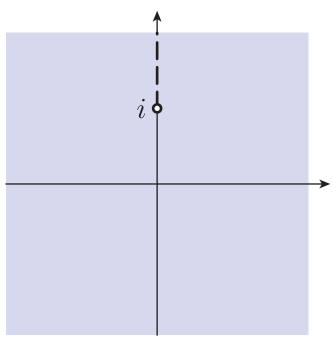
By the Chain Rule,

$$f'(z) = i \operatorname{Log}'(1 + iz) = \frac{i}{1 + iz}.$$

To determine the domain of f', notice that the conditions of the Chain Rule are fulfilled at α provided that Log is differentiable at $1 + i\alpha$, that is, provided that $1 + i\alpha$ does not belong to the real interval $(-\infty, 0]$. Now

$$\begin{aligned} 1 + i\alpha \in (-\infty, 0] &\iff i\alpha \in (-\infty, -1] \\ &\iff \alpha \in \{iy : y \geq 1\}. \end{aligned}$$

Therefore $f'(\alpha)$ exists provided that α does not belong to the set $\{iy : y \geq 1\}$.



If α does belong to $\{iy : y \geq 1\}$, then $f'(\alpha)$ cannot exist because f is discontinuous at such values of α.

It follows that the domain of f' is $\mathbb{C} - \{iy : y \geq 1\}$.

Solution to Exercise 3.5

The principal power functions in (a) and (b) can be differentiated using the corollary on derivatives of principal powers.

(a) $f'(z) = \pi z^{\pi - 1}$

(b) $f'(z) = \frac{3}{2} z^{1/2}$

The domain of both these derivatives is the cut plane $\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$.

By contrast, the integer power functions in (c) and (d) can be differentiated using the rule for differentiating integer powers given in Subsection 1.2.

(c) $f'(z) = 5z^4 \quad (z \in \mathbb{C})$

(d) $f'(z) = -3z^{-4} \quad (z \in \mathbb{C} - \{0\})$

Solution to Exercise 3.6

(a) By the Combination Rules,

$$\begin{aligned} f'(z) &= \left(e^z \times \frac{1}{z}\right) + (e^z \times \operatorname{Log} z) \\ &= \left(\frac{1}{z} + \operatorname{Log} z\right) e^z. \end{aligned}$$

Constant functions and the exponential function are entire, but the domain of the derivative of Log is $\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$. So this is also the domain of f'.

(b) By the Chain Rule,

$$f'(z) = 20(z + \sin z)^{19}(1 + \cos z).$$

Since the functions $z \longmapsto z + \sin z$ and $z \longmapsto z^{20}$ are both entire, we see that the domain of f' is $\mathbb{C}$.

(c) By the Chain Rule,

$$\begin{aligned} f'(z) &= 2\cos(z + \cosh z) \times (-\sin(z + \cosh z)) \\ &\qquad \times (1 + \sinh z) \\ &= -(\sin(2z + 2\cosh z))(1 + \sinh z). \end{aligned}$$

Since the functions $z \longmapsto z + \cosh z$ and $z \longmapsto \cos^2 z$ are both entire, we see that the domain of f' is $\mathbb{C}$.

(d) By the Quotient Rule,

$$\begin{aligned} f'(z) &= \frac{(z \times 1/z) - (\operatorname{Log} z \times 1)}{z^2} \\ &= \frac{1 - \operatorname{Log} z}{z^2}. \end{aligned}$$

Since the domain of the derivative of Log is $\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$, we see that this is also the domain of the derivative of f'.

With practice, the rules for differentiating can be applied by inspection, and in future we do not always provide details of which rules we have applied.

Solution to Exercise 3.7

(a) By the Reciprocal Rule,

$$\operatorname{sech}' z = -\frac{\sinh z}{\cosh^2 z} = -\operatorname{sech} z \tanh z.$$

(b) By the Reciprocal Rule,

$$\operatorname{cosech}' z = -\frac{\cosh z}{\sinh^2 z} = -\operatorname{cosech} z \coth z.$$

(c) By the Quotient Rule,

$$\begin{aligned} \coth' z &= \frac{\sinh^2 z - \cosh^2 z}{\sinh^2 z} \\ &= -\frac{1}{\sinh^2 z} = -\operatorname{cosech}^2 z. \end{aligned}$$

Solution to Exercise 3.8

The function $z \longmapsto z + 1$ is entire with derivative $z \longmapsto 1$. Since Log is differentiable on the cut plane $\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$, and $z + 1$ lies in the cut plane whenever $z \in \mathbb{C} - \{x \in \mathbb{R} : x \leq -1\}$, it follows that $z \longmapsto \operatorname{Log}(z + 1)$ is differentiable on $\mathbb{C} - \{x \in \mathbb{R} : x \leq -1\}$, with derivative $z \longmapsto \dfrac{1}{z+1}$, by the Chain Rule.

Since exp is entire, $z \longmapsto \exp(z^2)$ is differentiable on $\mathbb{C}$ with derivative $z \longmapsto 2z\exp(z^2)$, by the Chain Rule.

It follows from the Sum Rule that f is differentiable on $\mathbb{C} - \{x \in \mathbb{R} : x \leq -1\}$, with derivative

$$f'(z) = \frac{1}{z+1} + 2z\exp(z^2).$$

Now, f is not differentiable on $\{x \in \mathbb{R} : x \leq -1\}$ because it is discontinuous at points of this set. The domain of f' is therefore $\mathbb{C} - \{x \in \mathbb{R} : x \leq -1\}$.

(Note that this domain does not include all points at which $\dfrac{1}{z+1} + 2z\exp(z^2)$ is defined.)

Solution to Exercise 3.9

(a) $f'(z) = -(\sin z + 1) \times \cos(\cos z - z)$

(b) $f'(z) = 2z^2 \exp(z^2 + i) + \exp(z^2 + i) - \sin z$

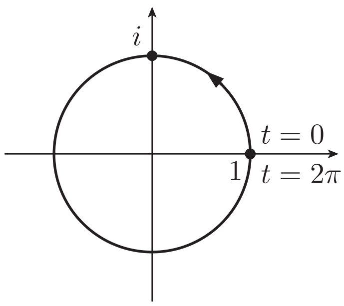
(c)
$$f'(z) = \frac{\exp(z+1)(1/z) - (\operatorname{Log} z)\exp(z+1)}{(\exp(z+1))^2} = \frac{(1/z) - \operatorname{Log} z}{\exp(z+1)}$$

(d)
$$f'(z) = -\sin(3\sin(i + \tan z)) \times 3\cos(i + \tan z) \times \sec^2 z$$

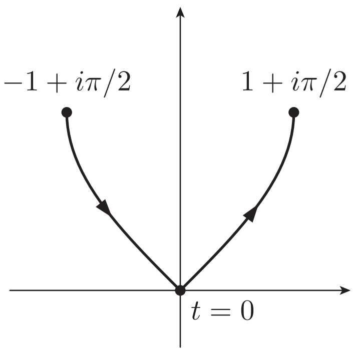
Each of these derivatives was established using the Combination Rules and the Chain Rule (part (d) used the extended form of the Chain Rule).

Solution to Exercise 4.1

(a)



(b)



Solution to Exercise 4.2

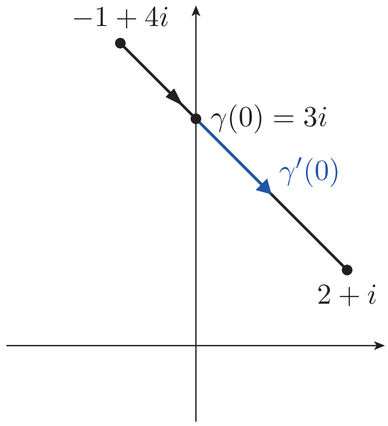
In each case we treat t as if it were the complex variable z.

(a) Using the Restriction Rule, we obtain

$$\gamma'(t) = 1 - i \quad (t \in [-1, 2]).$$

In other words, the derivative is constant throughout the interval $[-1, 2]$.

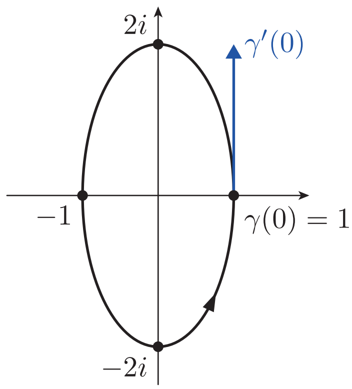
In particular, $\gamma'(0) = 1 - i$.



(b) Using the Restriction Rule, we obtain

$$\begin{aligned}\gamma'(t) &= \cos' t + 2i \sin' t \\ &= -\sin t + 2i \cos t \quad (t \in [0, 2\pi]).\end{aligned}$$

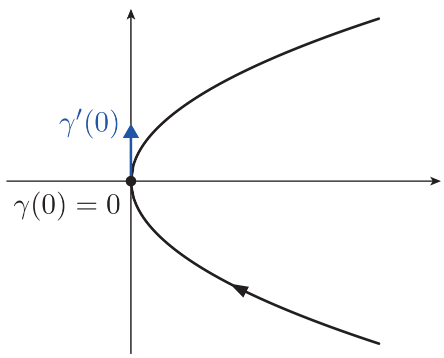
In particular, $\gamma'(0) = 2i$.



(c) Using the Restriction Rule, we obtain

$$\gamma'(t) = 2t + 2i \quad (t \in \mathbb{R}).$$

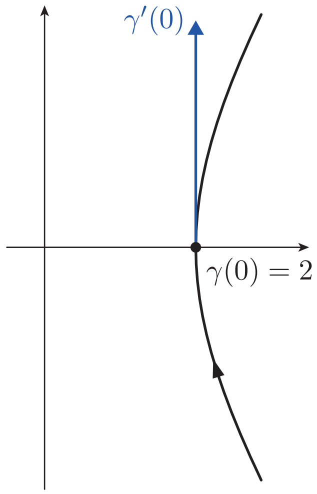
In particular, $\gamma'(0) = 2i$.



(d) Using the Restriction Rule, we obtain

$$\begin{aligned}\gamma'(t) &= 2\cosh' t + 3i \sinh' t \\ &= 2\sinh t + 3i \cosh t \quad (t \in \mathbb{R}).\end{aligned}$$

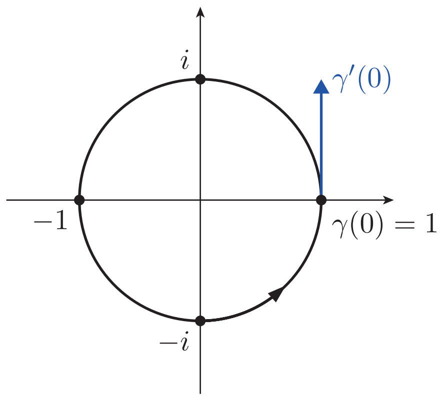
In particular, $\gamma'(0) = 3i$.



(e) Using the Restriction Rule, we obtain

$$\gamma'(t) = ie^{it} \quad (t \in [0, 2\pi]).$$

In particular, $\gamma'(0) = i$.



Solution to Exercise 4.3

$\operatorname{Re} \gamma(t) = t$ is differentiable on $[0, \infty)$, whereas $\operatorname{Im} \gamma(t) = \sqrt{t}$ is differentiable on $(0, \infty)$, but not at 0. To see that it is not differentiable at 0, observe that if $z_n = 1/n$, $n = 1, 2, \ldots$, then $z_n \to 0$ and

$$\frac{\sqrt{z_n} - \sqrt{0}}{z_n - 0} = \frac{1/\sqrt{n}}{1/n} = \sqrt{n} \to \infty \text{ as } n \to \infty.$$

It follows from Theorem 4.1 that γ is differentiable on $(0, \infty)$, but not at 0.

Solution to Exercise 4.4

(a) This parametrisation is differentiable and

$$\gamma'(t) = 1 + i \sin t \quad (t \in [0, 2\pi]).$$

Since $\gamma'(t)$ is continuous and non-zero on $[0, 2\pi]$, the parametrisation is smooth.

(b) Here

$$\gamma'(t) = 2t - 2 \quad (t \in [0, 2]),$$

so $\gamma'(1) = 0$. Hence γ is *not* smooth.

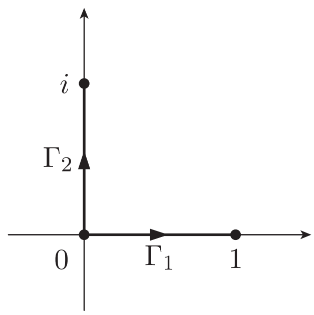
(c) The parametrisation is not differentiable when $t = 0$, so it is not smooth.

Solution to Exercise 4.5

Consider the paths Γ_1 and Γ_2 with parametrisations

$$\gamma_1(t) = t \quad (t \in [0, 1]),$$
$$\gamma_2(t) = it \quad (t \in [0, 1]).$$

These paths are straight lines that intersect at right angles.



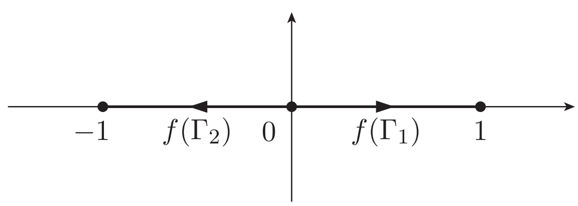
Under $f(z) = z^2$ they map to the paths with parametrisations

$$(f \circ \gamma_1)(t) = t^2 \quad (t \in [0, 1])$$

and

$$(f \circ \gamma_2)(t) = -t^2 \quad (t \in [0, 1]).$$

These image paths do not meet at right angles, since they both lie in the real axis.



The solution above does not contradict Theorem 4.2 because, although $f(z) = z^2$ is analytic at 0, $f'(0) = 0$.

Solution to Exercise 4.6

(a) Using the Restriction Rule, we obtain

$$\gamma'(t) = (2t + i)\exp(t^2 + it) \quad (t \in [-1, 1]).$$

(b) Using the Restriction Rule, we obtain

$$\gamma'(t) = 2t - i\sin t \quad (t \in [-1, 1]).$$

(c) Using the Restriction Rule, we obtain

$$\gamma'(t) = -5\sin t + 7i\cos t \quad (t \in [-\pi, \pi]).$$

Solution to Exercise 4.7

(a) This parametrisation is differentiable on the whole of its domain $I = [-1, 1]$. Furthermore, the derivative is continuous and non-zero on I, so the parametrisation is smooth.

(b) The derivative of this parametrisation is zero when $t = 0$, so the parametrisation is not smooth.

(c) This parametrisation is differentiable on the whole of its domain $I = [-\pi, \pi]$. Furthermore, the derivative is continuous and non-zero on I, so the parametrisation is smooth.

Solution to Exercise 4.8

(a) The paths meet at i because $\gamma_1(0) = \gamma_2(1) = i$. Now

$$\begin{aligned}\gamma_1'(t) &= 2\cos t + i,\\ \gamma_2'(t) &= -1 + 2it,\end{aligned}$$

so

$$\gamma_1'(0) = 2 + i \quad \text{and} \quad \gamma_2'(1) = -1 + 2i.$$

The angle from Γ_1 to Γ_2 at i is given by

$$\begin{aligned}\operatorname{Arg}\left(\frac{\gamma_2'(1)}{\gamma_1'(0)}\right) &= \operatorname{Arg}\left(\frac{-1+2i}{2+i}\right)\\ &= \operatorname{Arg}\left(\tfrac{1}{5}(-1+2i)(\overline{2+i})\right)\\ &= \operatorname{Arg}\left(\tfrac{1}{5}(-1+2i)(2-i)\right)\\ &= \operatorname{Arg} i = \pi/2.\end{aligned}$$

(b) Since $f'(z) = 2\sin z\cos z = \sin 2z$, we have

$$f'(i) = \sin 2i = i\sinh 2 \neq 0.$$

Hence f is conformal at i (by Theorem 4.2), so the angle from $f(\Gamma_1)$ to $f(\Gamma_2)$ at $f(i)$ is the same as the angle from Γ_1 to Γ_2 at i, namely $\pi/2$.

References

Bottazzini, U. and Gray, J. (2013) *Hidden Harmony – Geometric Fantasies: The Rise of Complex Function Theory*, Sources and Studies in the History of Mathematics and Physical Sciences, New York, Springer.

Burton, D. M. (2002) *The History of Mathematics: An Introduction*, New York, McGraw-Hill Companies.

Katz, V. J. (1993) *A History of Mathematics*, New York, HarperCollins College Publishers.

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